CHAPTER

14

LOCUS AND CONSTRUCTION

Chapter Table of Contents

Classical Greek construction problems limit the solution of the problem to the use of two instruments: the straightedge and the compass. There are three construction problems that have challenged mathematicians through the centuries and have been proved impossible:

- the duplication of the cube
- the trisection of an angle
- the squaring of the circle

The duplication of the cube requires that a cube be constructed that is equal in volume to twice that of a given cube. The origin of this problem has many versions. For example, it is said to stem from an attempt at Delos to appease the god Apollo by doubling the size of the altar dedicated to Apollo.

The trisection of an angle, separating the angle into three congruent parts using only a straightedge and compass, has intrigued mathematicians through the ages.

The squaring of the circle means constructing a square equal in area to the area of a circle. This is equivalent to constructing a line segment whose length is equal to $\sqrt{\pi}$ times the radius of the circle.

Although solutions to these problems have been presented using other instruments, solutions using only straightedge and compass have been proven to be impossible.

14-1 CONSTRUCTING PARALLEL LINES

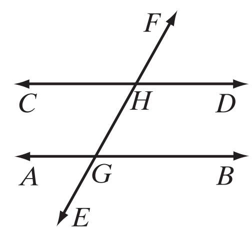
In Chapter 5, we developed procedures to construct the following lines and rays:

1. a line segment congruent to a given line segment
2. an angle congruent to a given angle
3. the bisector of a given line segment
4. the bisector of a given angle
5. a line perpendicular to a given line through a given point on the line
6. a line perpendicular to a given line through a given point not on the line

Then, in Chapter 9, we constructed parallel lines using the theorem that if two coplanar lines are each perpendicular to the same line, then they are parallel.

Now we want to use the construction of congruent angles to construct parallel lines.

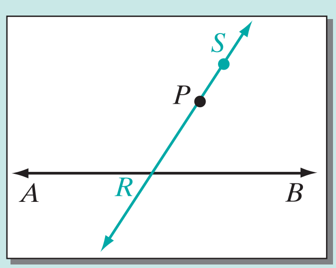
Two lines cut by a transversal are parallel if and only if the corresponding angles are congruent. For example, in the diagram, the transversal $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at G and $\overleftrightarrow{CD}$ at H. If $\angle EGB \cong \angle GHD$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Therefore, we can construct parallel lines by constructing congruent corresponding angles.



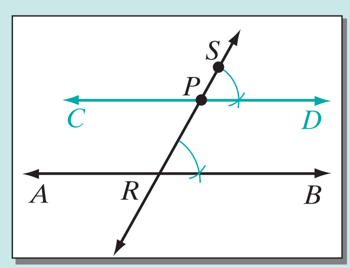
Construction 7 — Construct a Line Parallel to a Given Line at a Given Point.

Given $\overleftrightarrow{AB}$ and point P not on $\overleftrightarrow{AB}$

Construct A line through P that is parallel to $\overleftrightarrow{AB}$



STEP 1. Through P, draw any line intersecting $\overleftrightarrow{AB}$ at R. Let S be any point on the ray opposite $\overrightarrow{PR}$.



STEP 2. At P, construct $\angle SPD \cong \angle PRB$. Draw $\overrightarrow{PC}$, the opposite ray of $\overrightarrow{PD}$, forming $\overleftrightarrow{CD}$.

Continued

Construction 7 — Construct a Line Parallel to a Given Line at a Given Point. (*continued*)

Conclusion $\overleftrightarrow{CPD} \parallel \overleftrightarrow{AB}$

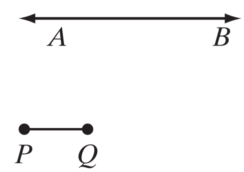
Proof Corresponding angles, $\angle SPD$ and $\angle PRB$, are congruent. Therefore, $\overleftrightarrow{CPD} \parallel \overleftrightarrow{AB}$.

This construction can be used to construct the points and lines that satisfy other conditions.

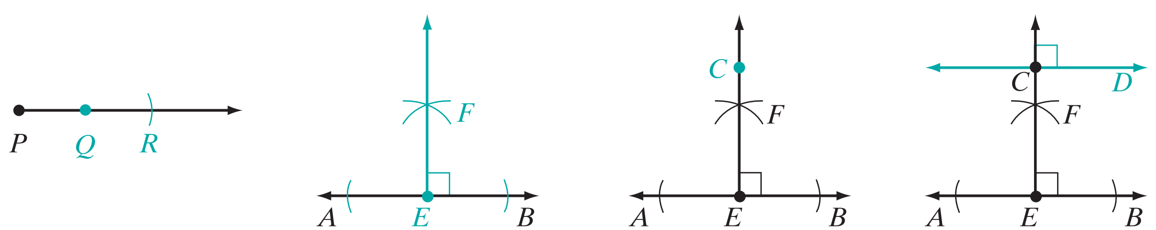
EXAMPLE 1

Given: $\overleftrightarrow{AB}$ and $\overline{PQ}$

Construct: $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ at a distance $2PQ$ from $\overleftrightarrow{AB}$.



Construction The distance from a point to a line is the length of the perpendicular from the point to the line. Therefore, we must locate points at a distance of $2PQ$ from a point on $\overleftrightarrow{AB}$ at which to draw a parallel line.



1. Extend $\overline{PQ}$. Locate point R on $\overrightarrow{PQ}$ such that $QR = PQ$, making $PR = 2PQ$.

2. Choose any point E on $\overleftrightarrow{AB}$. At E, construct the line $\overrightarrow{EF}$ perpendicular to $\overleftrightarrow{AB}$.

3. On $\overleftrightarrow{EF}$, locate point C at a distance PR from E.

4. At C, construct $\overleftrightarrow{CD}$ perpendicular to $\overleftrightarrow{EF}$ and therefore parallel to $\overleftrightarrow{AB}$.

Conclusion $\overleftrightarrow{CD}$ is parallel to $\overleftrightarrow{AB}$ at a distance $2PQ$ from $\overleftrightarrow{AB}$.

Exercises

Writing About Mathematics

1. In the example, every point on $\overleftrightarrow{CD}$ is at a fixed distance, $2PQ$, from $\overleftrightarrow{AB}$. Explain how you know that this is true.

2. Two lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, are parallel. A third line, $\overleftrightarrow{EF}$, is perpendicular to $\overleftrightarrow{AB}$ at G and to $\overleftrightarrow{CD}$ at H. The perpendicular bisector of $\overline{GH}$ is the set of all points equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Explain how you know that this is true.

Developing Skills

In 3–9, complete each required construction using a compass and straightedge, or geometry software. Draw each given figure and do the required constructions. Draw a separate figure for each construction. Enlarge the given figure for convenience.

3. *Given*: Parallel lines l and m and point P.

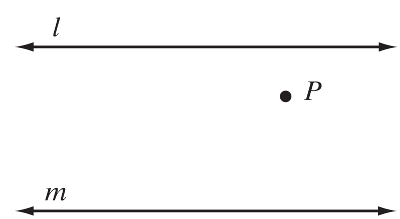
Construct:

a. a line through P that is parallel to l.

b. a line that is parallel to l and to m and is equidistant from l and m.

c. a line n that is parallel to l and to m such that l is equidistant from m and n.

d. Is the line that you constructed in **a** parallel to m? Justify your answer.



4. *Given*: Line segments of length a and b and $\angle A$

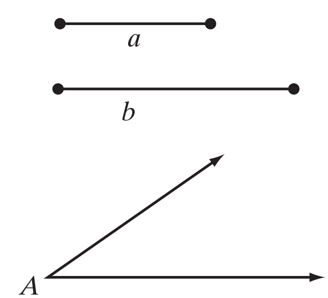
Construct:

a. a rectangle whose length is a and whose width is b.

b. a square such that the length of a side is a.

c. a parallelogram that has sides with measures a and b and an angle congruent to $\angle A$.

d. a rhombus that has sides with measure a and an angle congruent to $\angle A$.



5. *Given*: $\overline{AB}$

a. Divide $\overline{AB}$ into four congruent parts.

b. Construct a circle whose radius is AB.

c. Construct a circle whose diameter is AB.

6. *Given*: $\triangle ABC$

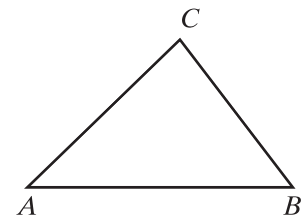
Construct:

a. a line parallel to $\overleftrightarrow{AB}$ at C.

b. the median to $\overline{AC}$ by first constructing the midpoint of $\overline{AC}$.

c. the median to $\overline{BC}$ by first constructing the midpoint of $\overline{BC}$.

d. If the median to $\overline{AC}$ and the median to $\overline{BC}$ intersect at P, can the median to $\overline{AB}$ be drawn without first locating the midpoint of $\overline{AB}$? Justify your answer.



7. *Given*: Obtuse triangle ABC

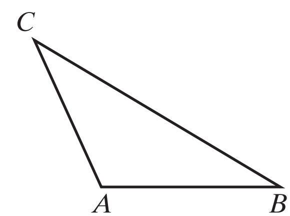
Construct:

a. the perpendicular bisector of $\overline{AC}$.

b. the perpendicular bisector of $\overline{BC}$.

c. M, the midpoint of $\overline{AB}$.

d. If the perpendicular bisector of $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ intersect at P, what two points determine the perpendicular bisector of $\overline{AB}$? Justify your answer.



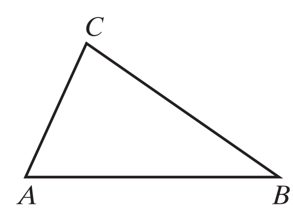
8. *Given*: $\triangle ABC$

Construct:

a. the altitude to $\overline{AC}$.

b. the altitude to $\overline{BC}$.

c. If the altitude to $\overline{AC}$ and the altitude to $\overline{BC}$ intersect at P, what two points determine the altitude to $\overline{AB}$? Justify your answer.



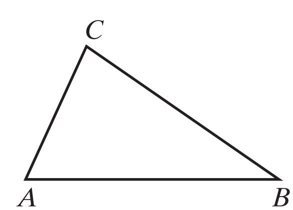
9. *Given*: $\triangle ABC$

Construct:

a. the angle bisector from B to $\overline{AC}$.

b. the angle bisector from A to $\overline{BC}$.

c. If the angle bisector of $\angle B$ and the angle bisector of $\angle A$ intersect at P, what two points determine the angle bisector from C to $\overline{AB}$? Justify your answer.



10. a. Draw any line segment, $\overline{AB}$. Draw any ray, $\overrightarrow{AC}$, forming $\angle BAC$.

b. Choose any point, D, on $\overrightarrow{AC}$. Construct $\overline{DE}$ on $\overrightarrow{AC}$ such that $DE = 2AD$.

c. Draw $\overline{EB}$ and construct a line through D parallel to $\overline{EB}$ and intersecting $\overrightarrow{AB}$ at F.

d. Prove that $AD : DE = AF : FB = 1 : 2$.

11. a. Draw an angle, $\angle LPR$.

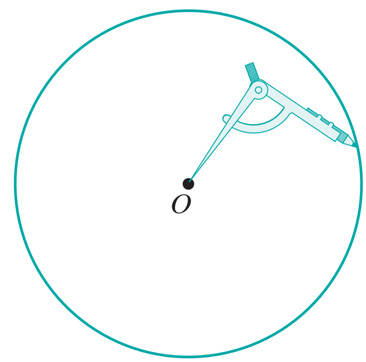
b. Choose point N on $\overrightarrow{PL}$. Construct Q on $\overrightarrow{PL}$ such that $PQ = 8PN$. Locate S on $\overline{PQ}$ such that $PS : SQ = 3 : 5$.

c. Draw $\overline{QR}$ and construct a line through S parallel to $\overline{QR}$ and intersecting $\overrightarrow{PR}$ at T.

d. Prove that $\triangle PST \sim \triangle PQR$ with a constant of proportionality of $\frac{3}{8}$.

14-2 THE MEANING OF LOCUS

In a construction, the opening between the pencil and the point of the compass is a fixed distance, the length of the radius of a circle. The point of the compass determines a fixed point, point O in the diagram. If the length of the radius remains unchanged, all of the points in the plane that can be drawn by the compass form a circle, and any points that cannot be drawn by the compass do not lie on the circle. Thus, the circle is the set of all points at a fixed distance from a fixed point. This set is called a *locus*.



DEFINITION

A **locus** is the set of all points that satisfy a given condition or set of conditions.

The example of the circle given above helps us to understand what the definition means. Every definition can be written as a biconditional:

p: A point is on the locus.
q: A point satisfies the given conditions.

1. $(p \rightarrow q)$: If a point is on the locus, then the point satisfies the given conditions. All points on the circle are at a given fixed distance from the center.

2. $(\sim p \rightarrow \sim q)$: If a point is not on the locus, then the point does not satisfy the given conditions. Any point that is *not* on the circle is *not* at the given distance from the center.

Recall that the statement $(\sim p \rightarrow \sim q)$ is the inverse of the statement $(p \rightarrow q)$. A locus is correct when both statements are true: the conditional and its inverse.

We can restate the definition of locus in biconditional form:

► **A point P is a point of the locus if and only if P satisfies the given conditions of the locus.**

Discovering a Locus

Procedure

To discover a probable locus:

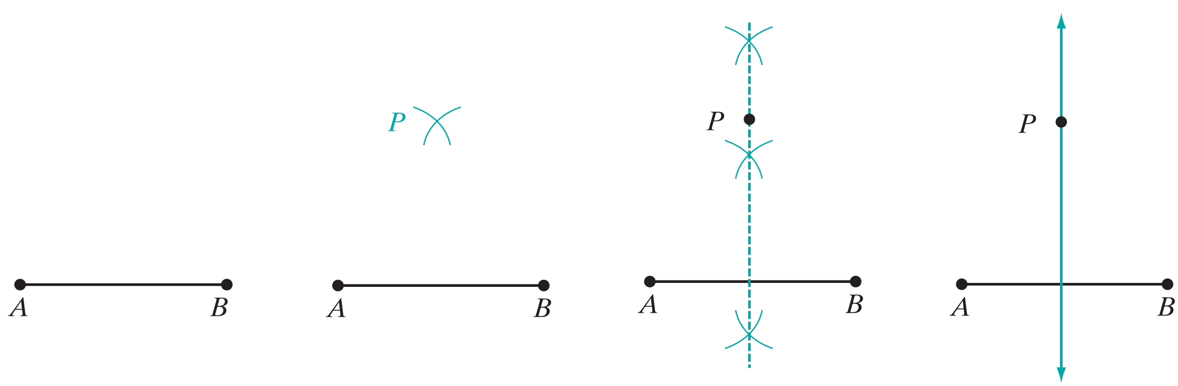
1. Make a diagram that contains the fixed lines or points that are given.
2. Decide what condition must be satisfied and locate one point that meets the given condition.
3. Locate several other points that satisfy the given condition. These points should be sufficiently close together to develop the shape or the nature of the locus.
4. Use the points to draw a line or smooth curve that appears to be the locus.
5. Describe in words the geometric figure that appears to be the locus.

Note: In this chapter, we will assume that all given points, segments, rays, lines, and circles lie in the same plane and the desired locus lies in that plane also.

EXAMPLE 1

What is the locus of points equidistant from the endpoints of a given line segment?

Solution Apply the steps of the procedure for discovering a probable locus.



1. Make a diagram: $\overline{AB}$ is the given line segment.

2. Decide the condition to be satisfied: P is to be equidistant from A and B. Use a compass opened to any convenient radius to locate one such point, P.

3. Locate several other points equidistant from A and B, using a different opening of the compass for each point.

4. Through these points, draw the straight line that appears to be the locus.

5. Describe the locus in words: The locus is a straight line that is the perpendicular bisector of the given line segment. *Answer*

Note that in earlier chapters, we proved two theorems that justify these results:

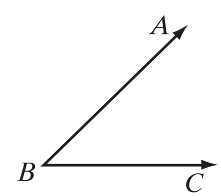
- If a point is equidistant from the endpoints of a line segment, then it lies on the perpendicular bisector of the segment.
- If a point lies on the perpendicular bisector of a line segment, then it is equidistant from the endpoints of the segment.

EXAMPLE 2

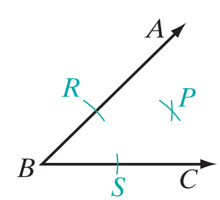
Construct the locus of points in the interior of an angle equidistant from the rays that form the sides of the given angle.

Construction Corollaries 9.13b and 9.15a together state: A point is equidistant from the sides of an angle if and only if it lies on the bisector of the angle. Therefore, the required locus is the bisector of the angle.

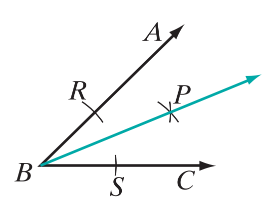
1. Make a diagram: $\angle ABC$ is the given angle.



2. Decide the condition to be satisfied: P is to be equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$, the rays that are the sides of $\angle ABC$. Construct the angle bisector. Use a compass to draw an arc with center B that intersects $\overrightarrow{BA}$ at R and $\overrightarrow{BC}$ at S. Then, with the compass open to a convenient radius, draw arcs from R and S that intersect in the interior of $\angle ABC$. Call the intersection P. $PR = PS$.



3. Through points P and B, draw the ray that is the locus.



Exercises

Writing About Mathematics

1. Are all of the points that are equidistant from the endpoints of a line segment that is 8 centimeters long 4 centimeters from the endpoints? Explain your answer.

2. What line segment do we measure to find the distance from a point to a line or to a ray?

Developing Skills

3. What is the locus of points that are 10 centimeters from a given point?

4. What is the locus of points equidistant from two points, A and B, that are 8 meters apart?

5. What is the locus of points equidistant from two parallel lines 8 meters apart?

6. What is the locus of points 4 inches away from a given line, $\overleftrightarrow{AB}$?

7. What is the locus of points 3 inches from each of two parallel lines that are 6 inches apart?

8. What is the locus of points that are equidistant from two opposite sides of a square?

9. What is the locus of points that are equidistant from the vertices of two opposite angles of a square?

10. What is the locus of points that are equidistant from the four vertices of a square? (A locus can consist of a single point or no points.)

11. What is the locus of points in the interior of a circle whose radius measures 3 inches if the points are 2 inches from the circle?

12. What is the locus of points in the exterior of a circle whose radius measures 3 inches if the points are 2 inches from the circle?

13. What is the locus of points 2 inches from a circle whose radius measures 3 inches?

14. Circle O has a radius of length r, and it is given that $r > m$.
    a. What is the locus of points in the exterior of circle O and at a distance m from the circle?
    b. What is the locus of points in the interior of circle O and at a distance m from the circle?
    c. What is the locus of points at a distance m from circle O?

15. **Concentric circles** have the same center. What is the locus of points equidistant from two concentric circles whose radii measure 10 centimeters and 18 centimeters?

16. A series of isosceles triangles are drawn, each of which has a fixed segment, $\overline{AB}$, as its base. What is the locus of the vertices of the vertex angles of all such isosceles triangles?

17. Triangle ABC is drawn with a fixed base, $\overline{AB}$, and an altitude to $\overline{AB}$ whose measure is 3 feet. What is the locus of points that can indicate vertex C in all such triangles?

Applying Skills

18. What is the locus of the tip of the hour hand of a clock during a 12-hour period?

19. What is the locus of the center of a train wheel that is moving along a straight, level track?

20. What is the locus of the path of a car that is being driven down a street equidistant from the two opposite parallel curbs?

21. A dog is tied to a stake by a rope 6 meters long. What is the boundary of the surface over which he can move?

22. A boy walks through an open field that is bounded on two sides by straight, intersecting roads. He walks so that he is always equidistant from the two intersecting roads. Determine his path.

23. There are two stationary floats on a lake. A girl swims so that she is always equidistant from both floats. Determine her path.

24. A dime is rolled along a horizontal line so that the dime always touches the line. What is the locus of the center of the dime?

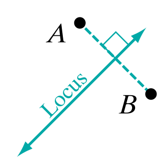
25. The outer edge of circular track is 40 feet from a central point. The track is 10 feet wide. What is path of a runner who runs on the track, 2 feet from the inner edge of the track?

14-3 FIVE FUNDAMENTAL LOCI

There are five fundamental loci, each based on a different set of conditions. In each of the following, a condition is stated, and the locus that fits the condition is described in words and drawn below. Each of these loci has been shown as a construction in preceding sections.

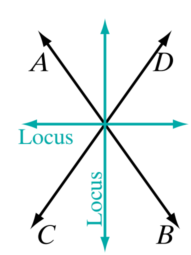
1. *Equidistant from two points:* Find points equidistant from points A and B.

Locus: The locus of points equidistant from two fixed points is the perpendicular bisector of the segment determined by the two points.



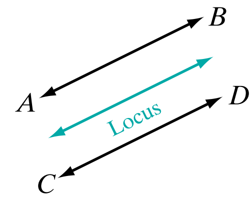
2. *Equidistant from two intersecting lines:* Find points equidistant from the intersecting lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Locus: The locus of points equidistant from two intersecting lines is a pair of lines that bisect the angles formed by the intersecting lines.



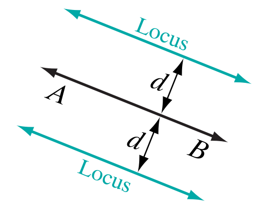
3. *Equidistant from two parallel lines:* Find points equidistant from the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Locus: The locus of points equidistant from two parallel lines is a third line, parallel to the given lines and midway between them.



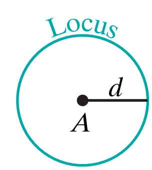
4. *At a fixed distance from a line:* Find points that are at distance d from the line $\overleftrightarrow{AB}$.

Locus: The locus of points at a fixed distance from a line is a pair of lines, each parallel to the given line and at the fixed distance from the given line.



5. *At a fixed distance from a point:* Find points that are at a distance d from the fixed point A.

Locus: The locus of points at a fixed distance from a fixed point is a circle whose center is the fixed point and whose radius is the fixed distance.



These loci are often combined to find a point or set of points that satisfy two or more conditions. The resulting set is called a **compound locus**.

Procedure

To find the locus of points that satisfy two conditions:

1. Determine the locus of points that satisfy the first condition. Sketch a diagram showing these points.
2. Determine the locus of points that satisfy the second condition. Sketch these points on the diagram drawn in step 1.
3. Locate the points, if any exist, that are common to both loci.

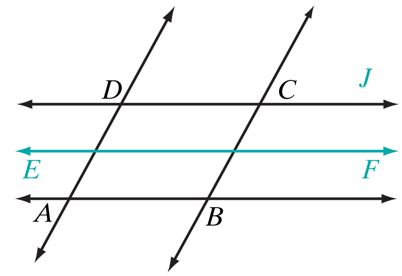
Steps 2 and 3 can be repeated if the locus must satisfy three or more conditions.

EXAMPLE 1

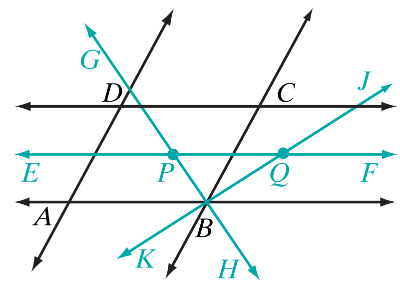
Quadrilateral $ABCD$ is a parallelogram. What is the locus of points equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and also equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$?

Solution Follow the procedure for finding a compound locus.

(1) Since $ABCD$ is a parallelogram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The locus of points equidistant from two parallel lines is a third line parallel to the given lines and midway between them. In the figure, $\overleftrightarrow{EF}$ is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.



(2) The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are intersecting lines. The locus of points equidistant from intersecting lines is a pair of lines that bisect the angles formed by the given lines. In the figure, $\overleftrightarrow{GH}$ and $\overleftrightarrow{JK}$ are equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.



(3) The point P at which $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{GH}$ and the point Q at which $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{JK}$ are equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and also equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.

Answer P and Q

Note that only point P is equidistant from the three segments that are sides of the parallelogram, but both P and Q are equidistant from the lines of which these three sides are segments.

Exercises

Writing About Mathematics

1. If $PQRS$ is a square, are the points that are equidistant from $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ also equidistant from P and S? Explain your answer.

2. Show that the two lines that are equidistant from two intersecting lines are perpendicular to each other.

Developing Skills

In 3–10, sketch and describe each required locus.

3. The locus of points equidistant from two points that are 4 centimeters apart.

4. The locus of points that are 6 inches from the midpoint of a segment that is 1 foot long.

5. The locus of points equidistant from the endpoints of the base of an isosceles triangle.

6. The locus of points equidistant from the legs of an isosceles triangle.

7. The locus of points equidistant from the diagonals of a square.

8. The locus of points equidistant from the lines that contain the bases of a trapezoid.

9. The locus of points 4 centimeters from the midpoint of the base of an isosceles triangle if the base is 8 centimeters long.

10. The locus of points that are 6 centimeters from the altitude to the base of an isosceles triangle if the measure of the base is 12 centimeters

11. **a.** Sketch the locus of points equidistant from two parallel lines that are 4 centimeters apart.

b. On the diagram drawn in **a**, place point P on one of the given parallel lines. Sketch the locus of points that are 3 centimeters from P.

c. How many points are equidistant from the two parallel lines and 3 centimeters from P?

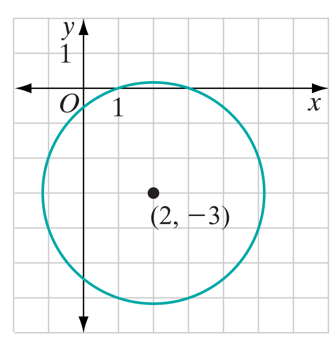
12. **a.** Construct the locus of points equidistant from the endpoints of a line segment $\overline{AB}$.

b. Construct the locus of points at a distance $\frac{1}{2}(AB)$ from M, the midpoint of $\overline{AB}$.

c. How many points are equidistant from A and B and at a distance $\frac{1}{2}(AB)$ from the midpoint of $\overline{AB}$?

d. Draw line segments joining A and B to the points described in **c** to form a polygon. What kind of a polygon was formed? Explain your answer.

14-4 POINTS AT A FIXED DISTANCE IN COORDINATE GEOMETRY



We know that the locus of points at a fixed distance from a given point is a circle whose radius is the fixed distance. In the coordinate plane, the locus of points r units from (h, k) is the circle whose equation is $(x - h)^2 + (y - k)^2 = r^2$.

For example, the equation of the locus of points $\sqrt{10}$ units from $(2, -3)$ is:

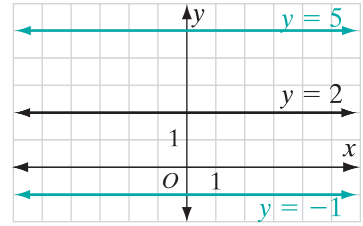
$$(x - 2)^2 + (y - (-3))^2 = (\sqrt{10})^2$$

or

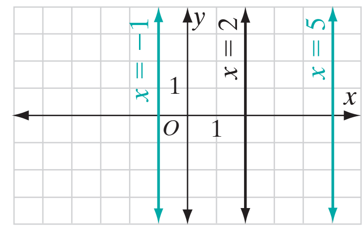
$$(x - 2)^2 + (y + 3)^2 = 10$$

We also know that the locus of points at a fixed distance from a given line is a pair of lines parallel to the given line.

For example, to write the equations of the locus of points 3 units from the horizontal line $y = 2$, we need to write the equations of two horizontal lines, one 3 units above the line $y = 2$ and the other 3 units below the line $y = 2$. The equations of the locus are $y = 5$ and $y = -1$.



To write the equations of the locus of points 3 units from the vertical line $x = 2$, we need to write the equations of two vertical lines, one 3 units to the right of the line $x = 2$ and the other 3 units to the left of the line $x = 2$. The equations of the locus are $x = 5$ and $x = -1$.



From these examples, we can infer the following:

- **The locus of points d units from the horizontal line $y = a$ is the pair of lines $y = a + d$ and $y = a - d$.**
- **The locus of points d units from the vertical line $x = a$ is the pair of lines $x = a + d$ and $x = a - d$.**

EXAMPLE 1

What is the equation of the locus of points at a distance of $\sqrt{20}$ units from (0, 1)?

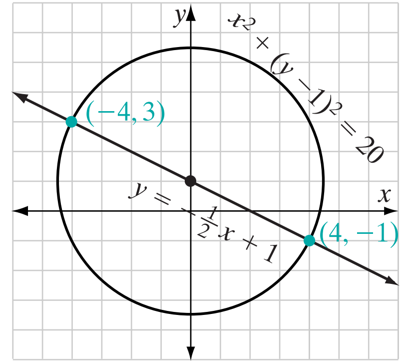
Solution The locus of points at a fixed distance from a point is the circle with the given point as center and the given distance as radius. The equation of the locus is

$$(x - 0)^2 + (y - 1)^2 = \left(\sqrt{20}\right)^2 \quad \text{or} \quad x^2 + (y - 1)^2 = 20$$ *Answer*

EXAMPLE 2

What are the coordinates of the points on the line $y = -\frac{1}{2}x + 1$ at a distance of $\sqrt{20}$ from (0, 1)?

Solution From Example 1, we know that the set of all points at a distance of $\sqrt{20}$ from (0, 1) lie on the circle whose equation is $x^2 + (y - 1)^2 = 20$. Therefore, the points on the line $y = -\frac{1}{2}x + 1$ at a distance of $\sqrt{20}$ from (0, 1) are the intersections of the circle and the line.



METHOD 1 Solve the system of equations graphically. Sketch the graphs and read the coordinates from the graph.

METHOD 2 Solve the system of equations algebraically. Substitute the value of y from the linear equation in the equation of the circle.

$$x^2 + (y - 1)^2 = 20$$
$$x^2 + \left(-\tfrac{1}{2}x + 1 - 1\right)^2 = 20$$
$$x^2 + \tfrac{1}{4}x^2 = 20$$
$$\tfrac{5}{4}x^2 = 20$$
$$x^2 = 16$$
$$x = \pm 4$$

If $x = 4$:

$$y = -\tfrac{1}{2}(4) + 1$$
$$y = -1$$

If $x = -4$:

$$y = -\tfrac{1}{2}(-4) + 1$$
$$y = 3$$

Answer The points are $(4, -1)$ and $(-4, 3)$.

Exercises

Writing About Mathematics

1. In Example 2, is the line $y = -\frac{1}{2}x + 1$ a secant of the circle $x^2 + (y - 1)^2 = 20$? Justify your answer.

2. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangent to circle O. Martin said that point O is in the locus of points equidistant from $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$. Do you agree with Martin? Explain why or why not.

Developing Skills

In 3–8, write an equation of the locus of points at the given distance d from the given point P.

3. $d = 4, P(0, 0)$ **4.** $d = 1, P(-1, 0)$ **5.** $d = 3, P(0, -2)$

6. $d = 7, P(1, 1)$ **7.** $d = \sqrt{10}, P(3, -1)$ **8.** $d = \sqrt{18}, P(-3, 5)$

In 9–12, find the equations of the locus of points at the given distance d from the given line.

9. $d = 5, x = 7$ **10.** $d = 1, x = 1$ **11.** $d = 4, y = 2$ **12.** $d = 6, y = 7$

In 13–16, find the coordinates of the points at the given distance from the given point and on the given line.

13. 5 units from $(0, 0)$ on $y = x + 1$

14. 13 units from $(0, 0)$ on $y = x - 7$

15. 10 units from $(0, 1)$ on $y = x + 3$

16. $\sqrt{10}$ units from $(1, -1)$ on $y = x + 2$

In 17–22, write the equation(s) or coordinates and sketch each locus.

17. a. The locus of points that are 3 units from $y = 4$.

b. The locus of points that are 1 unit from $x = 2$.

c. The locus of points that are 3 units from $y = 4$ and 1 unit from $x = 2$.

18. a. The locus of points that are 3 units from $(2, 2)$.

b. The locus of points that are 3 units from $y = -1$.

c. The locus of points that are 3 units from $(2, 2)$ and 3 units from $y = -1$.

19. a. The locus of points that are 5 units from the origin.

b. The locus of points that are 3 units from the x-axis.

c. The locus of points that are 5 units from the origin and 3 units from the x-axis.

20. a. The locus of points that are 10 units from the origin.

b. The locus of points that are 8 units from the y-axis.

c. The locus of points that are 10 units from the origin and 8 units from the y-axis.

21. a. The locus of points that are 2 units from $(x + 4)^2 + y^2 = 16$.

b. The locus of points that are 2 units from the y-axis.

c. The locus of points that are 2 units from $(x + 4)^2 + y^2 = 16$ and 2 units from the y-axis.

22. a. The locus of points that are 3 units from $(x + 1)^2 + (y - 5)^2 = 4$.

b. The locus of points that are $\frac{5}{2}$ units from $x = \frac{3}{2}$.

c. The locus of points that are 3 units from $(x + 1)^2 + (y - 5)^2 = 4$ and $\frac{5}{2}$ units from $x = \frac{3}{2}$.

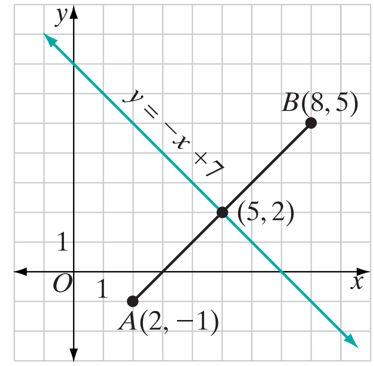
14-5 EQUIDISTANT LINES IN COORDINATE GEOMETRY

Equidistant from Two Points

The locus of points equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points. For example, the locus of points equidistant from $A(2, -1)$ and $B(8, 5)$ is a line perpendicular to $\overline{AB}$ at its midpoint.

$$\text{midpoint of } \overline{AB} = \left(\frac{2+8}{2}, \frac{5+(-1)}{2}\right) = (5, 2)$$

$$\text{slope of } \overline{AB} = \frac{5-(-1)}{8-2} = \frac{6}{6} = 1$$



Therefore, the slope of a line perpendicular to $\overline{AB}$ is -1. The perpendicular bisector of $\overline{AB}$ is the line through (5, 2) with slope -1. The equation of this line is:

$$\frac{y-2}{x-5} = -1$$
$$y - 2 = -x + 5$$
$$y = -x + 7$$

EXAMPLE 1

Describe and write an equation for the locus of points equidistant from $A(-2, 5)$ and $B(6, 1)$.

Solution (1) Find the midpoint, M, of $\overline{AB}$:

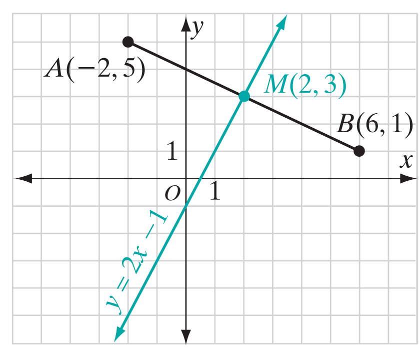
$$M = \left(\frac{-2+6}{2}, \frac{5+1}{2}\right) = (2, 3)$$

(2) Find the slope of $\overline{AB}$:

$$\text{slope of } \overline{AB} = \frac{5-1}{-2-6} = \frac{4}{-8} = -\frac{1}{2}$$

(3) The slope of a line perpendicular to $\overline{AB}$ is 2.

(4) Write an equation of the line through (2, 3) with slope 2.



$$\frac{y-3}{x-2} = 2$$
$$y - 3 = 2x - 4$$
$$y = 2x - 1$$

Answer The locus of points equidistant from $A(-2, 5)$ and $B(6, 1)$ is the perpendicular bisector of $\overline{AB}$. The equation of the locus is $y = 2x - 1$.

Equidistant from Two Parallel Lines

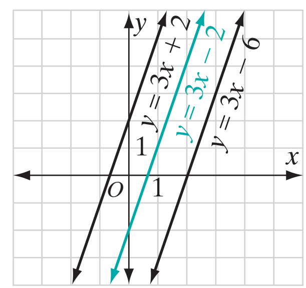
The locus of points equidistant from two parallel lines is a line parallel to the two lines and midway between them.

For example, the locus of points equidistant from the vertical lines $x = -2$ and $x = 6$ is a vertical line midway between them. Since the given lines intersect the x-axis at $(-2, 0)$ and $(6, 0)$, the line midway between them intersects the x-axis at $(2, 0)$ and has the equation $x = 2$.

EXAMPLE 2

Write an equation of the locus of points equidistant from the parallel lines $y = 3x + 2$ and $y = 3x - 6$.

Solution The locus is a line parallel to the given lines and midway between them.



The slope of the locus is 3, the slope of the given lines.

The y-intercept of the locus, b, is the average of the y-intercepts of the given lines.

$$b = \frac{2 + (-6)}{2} = \frac{-4}{2} = -2$$

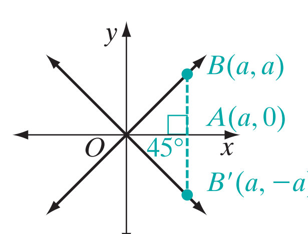
The equation of the locus is $y = 3x - 2$. *Answer*

Note that in Example 2, we have used the midpoint of the y-intercepts of the given lines as the y-intercept of the locus. In Exercise 21, you will prove that the midpoint of the segment at which the two given parallel lines intercept the y-axis is the point at which the line equidistant from the given lines intersects the y-axis.

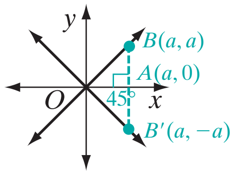
Equidistant from Two Intersecting Lines

The locus of points equidistant from two intersecting lines is a pair of lines that are perpendicular to each other and bisect the angles at which the given lines intersect. We will consider two special cases.

1. *The locus of points equidistant from the axes*



The x-axis and the y-axis intersect at the origin to form right angles. Therefore, the lines that bisect the angles between the axes will also go through the origin and will form angles measuring 45° with the axes. One bisector will have a positive slope and one will have a negative slope.



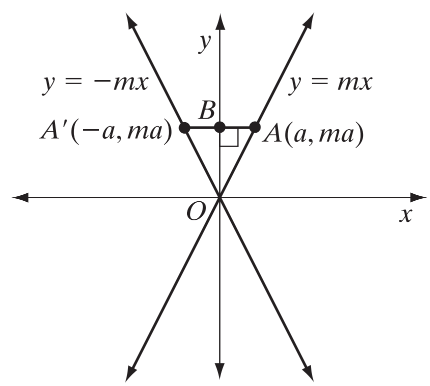
Let $A(a, 0)$ be a point on the x-axis and B be a point on the bisector with a positive slope such that $\overline{AB}$ is perpendicular to the x-axis. The triangle formed by A, B, and the origin O is a 45-45 right triangle. Since 45-45 right triangles are isosceles, $OA = AB = |a|$, and the coordinates of B are (a, a). The line through (a, a) and the origin is $y = x$.

Similarly, if B' is a point on the bisector with a negative slope, then the coordinates of B' are $(a, -a)$. The line through $(a, -a)$ and the origin is $y = -x$.

We have shown that the lines $y = x$ and $y = -x$ are the locus of points equidistant from the axes. These lines are perpendicular to each other since their slopes, 1 and -1, are negative reciprocals.

► **The locus of points in the coordinate plane equidistant from the axes is the pair of lines $y = x$ and $y = -x$.**

2. *The locus of points equidistant from two lines with slopes m and −m that intersect at the origin*



Let $O(0, 0)$ and $A(a, ma)$ be any two points on $y = mx$ and $B(0, ma)$ be a point on the y-axis. Under a reflection in the y-axis, the image of $O(0, 0)$ is $O(0, 0)$ and the image of $A(a, ma)$ is $A'(-a, ma)$. The points O and A' are on the line $y = -mx$. Therefore, the image of the line $y = mx$ is the line $y = -mx$ since collinearity is preserved under a line reflection. Also, $m\angle AOB = m\angle A'OB$ since angle measure is preserved under a line reflection. Therefore, the y-axis bisects the angle between the lines $y = mx$ and $y = -mx$. In a similar way, it can be shown that the x-axis bisects the other pair of angles between $y = mx$ and $y = -mx$. Therefore, the y-axis, together with the x-axis, is the locus of points equidistant from the lines $y = mx$ and $y = -mx$.

► **The locus of points in the coordinate plane equidistant from two lines with slopes m and $-m$ that intersect at the origin is the pair of lines $y = 0$ and $x = 0$, that is, the x-axis and the y-axis.**

Exercises

Writing About Mathematics

1. In the coordinate plane, are the x-axis and the y-axis the locus of points equidistant from the intersecting lines $y = x$ and $y = -x$? Justify your answer.

2. Ryan said that if the locus of points equidistant from $y = x + 2$ and $y = x + 10$ is $y = x + 6$, then the distance from $y = x + 6$ to $y = x + 2$ and to $y = x + 10$ is 4. Do you agree with Ryan? Justify your answer.

Developing Skills

In 3–8, find the equation of the locus of points equidistant from each given pair of points.

3. (1, 1) and (9, 1) **4.** (3, 1) and (3, −3) **5.** (0, 2) and (2, 0)

6. (0, 6) and (4, −2) **7.** (−2, −2) and (0, 2) **8.** (−4, −5) and (−2, 1)

In 9–14, find the equation of the locus of points equidistant from each given pair of parallel lines

9. $x = -1$ and $x = 7$ **10.** $y = 0$ and $y = -6$

11. $y = x + 3$ and $y = x + 9$ **12.** $y = -x - 2$ and $y = -x + 6$

13. $y = 2x + 1$ and $y = 2x + 5$ **14.** $2x + y = 7$ and $y = -2x + 9$

15. Find the coordinates of the locus of points equidistant from (−2, 3) and (4, 3), and 3 units from (1, 3).

16. Find the coordinates of the locus of points equidistant from (2, −5) and (2, 3), and 4 units from (0, −1).

Applying Skills

17. **a.** Find the equation of the locus of points equidistant from (3, 1) and (5, 5).

b. Prove that the point (−2, 6) is equidistant from the points (3, 1) and (5, 5) by showing that it lies on the line whose equation you wrote in **a**.

c. Prove that the point (−2, 6) is equidistant from (3, 1) and (5, 5) by using the distance formula.

18. **a.** Find the equation of the locus of points equidistant from the parallel lines $y = x + 3$ and $y = x - 5$.

b. Show that point $P(3, 2)$ is equidistant from the given parallel lines by showing that it lies on the line whose equation you wrote in **a**.

c. Find the equation of the line that is perpendicular to the given parallel lines through point $P(3, 2)$.

d. Find the coordinates of point A at which the line whose equation you wrote in **c** intersects $y = x + 3$.

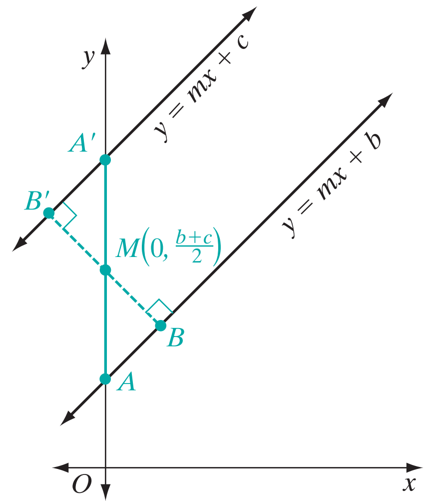
e. Find the coordinates of point B at which the line whose equation you wrote in **c** intersects $y = x - 5$.

f. Use the distance formula to show that $PA = PB$, that is, that P is equidistant from $y = x + 3$ and $y = x - 5$.

19. Show that the locus of points equidistant from the line $y = x + 1$ and the line $y = -x + 1$ is the y-axis and the line $y = 1$.

20. Show that the locus of points equidistant from the line $y = 3x - 2$ and the line $y = -3x - 2$ is the y-axis and the line $y = -2$.

21. Prove that the midpoint of the segment at which two given parallel lines intercept the y-axis is the point at which the line equidistant from the given lines intersects the y-axis. That is, if the y-intercept of the first line is b, and the y-intercept of the second line is c, then the y-intercept of the line equidistant from them is $\frac{b + c}{2}$.



a. Find the coordinates of A, the point where the first line intersects the y-axis.

b. Find the coordinates of A', the point where the second line intersects the y-axis.

c. Show that $M\left(0, \frac{b + c}{2}\right)$ is the midpoint of $\overline{AA'}$.

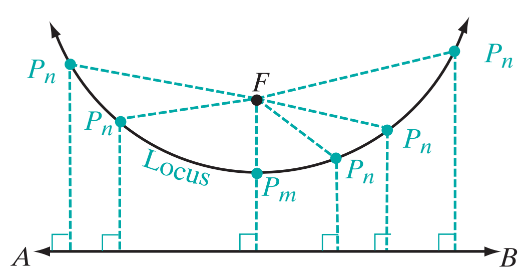
d. Let B and B' be points on the lines such that M is a point on $\overline{BB'}$ and $\overline{BB'}$ is perpendicular to both lines. Show that $\triangle ABM \cong \triangle A'B'M$.

e. Using part **d**, explain why M is the point at which the line equidistant from the given lines intersects the y-axis.

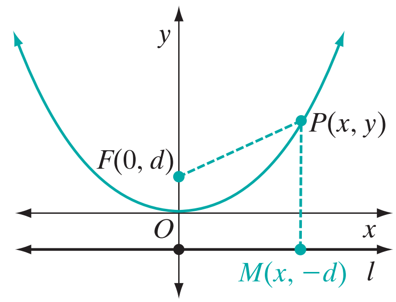
14-6 POINTS EQUIDISTANT FROM A POINT AND A LINE

We have seen that a straight line is the locus of points equidistant from two points or from two parallel lines. What is the locus of points equidistant from a given point and a given line?

Consider a fixed horizontal line $\overleftrightarrow{AB}$ and a fixed point F above the line. The point P_m, that is, the midpoint of the vertical line from F to $\overleftrightarrow{AB}$, is on the locus of points equidistant from the point and the line. Let P_n be any other point on the locus. As we move to the right or to the left from P_m along $\overleftrightarrow{AB}$, the distance from $\overleftrightarrow{AB}$ to P_n continues to be the length of a vertical line, but the distance from F to P_n is along a slant line. The locus of points is a curve.



Consider a point and a horizontal line that are at a distance d from the origin.



- Let $F(0, d)$ be the point and l be the line. The equation of l is $y = -d$.
- Let $P(x, y)$ be any point equidistant from F and l.
- Let $M(x, -d)$ be the point at which a vertical line from P intersects l.

The distance from P to M is equal to the distance from P to F.

$$\begin{aligned} PM &= PF \\ |y - (-d)| &= \sqrt{(x-0)^2 + (y-d)^2} \\ (y+d)^2 &= x^2 + (y-d)^2 \\ y^2 + 2dy + d^2 &= x^2 + y^2 - 2dy + d^2 \\ 4dy &= x^2 \end{aligned}$$

For instance, if $d = \frac{1}{4}$, that is, if the given point is $\left(0, \frac{1}{4}\right)$ and the given line is $y = -\frac{1}{4}$, then the equation of the locus is $y = x^2$. Recall that $y = x^2$ is the equation of a parabola whose turning point is the origin and whose axis of symmetry is the y-axis.

Recall that under the translation $T_{h,k}$, the image of $4dy = x^2$ is

$$4d(y-k) = (x-h)^2.$$

For example, if the fixed point is $F(2, 1)$ and the fixed line is $y = -1$, then $d = 1$ and the turning point of the parabola is $(2, 0)$, so $(h, k) = (2, 0)$. Therefore, the equation of the parabola is

$$4(1)(y-0) = (x-2)^2 \qquad \text{or} \qquad 4y = x^2 - 4x + 4.$$

This equation can also be written as $y = \frac{1}{4}x^2 - x + 1$.

For any horizontal line and any point not on the line, the equation of the locus of points equidistant from the point and the line is a parabola whose equation can be written as $y = ax^2 + bx + c$. If the given point is above the line, the coefficient a is positive and the parabola opens upward. If the given point is below the line, the coefficient a is negative and the parabola opens downward. The axis of symmetry is a vertical line whose equation is $x = \frac{-b}{2a}$. Since the turning point is on the axis of symmetry, its x-coordinate is $\frac{-b}{2a}$.

EXAMPLE 1

a. Draw the graph of $y = x^2 - 4x - 1$ from $x = -1$ to $x = 5$.

b. Write the coordinates of the turning point.

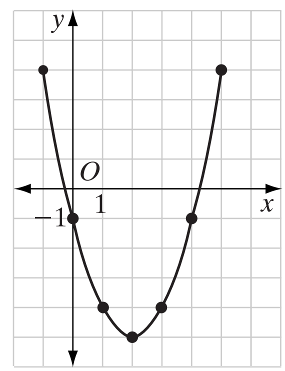
c. Write an equation of the axis of symmetry.

d. What are the coordinates of the fixed point and the fixed line for this parabola?

Solution **a.** (1) Make a table of values using integral values of x from $x = -1$ to $x = 5$.

(2) Plot the points whose coordinates are given in the table and draw a smooth curve through them.

x	$x^2 - 4x - 1$	y
−1	1 + 4 − 1	4
0	0 − 0 − 1	−1
1	1 − 4 − 1	−4
2	4 − 8 − 1	−5
3	9 − 12 − 1	−4
4	16 − 16 − 1	−1
5	25 − 20 − 1	4



b. From the graph or from the table, the coordinates of the turning point appear to be (2, −5). We can verify this algebraically:

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2$$

$$y = x^2 - 4x - 1 = (2)^2 - 4(2) - 1 = -5$$

c. The axis of symmetry is the vertical line through the turning point, $x = 2$.

d. Note that the turning point of the parabola is (2, −5). When the turning point of the parabola $y = x^2$ has been moved 2 units to the right and 5 units down, the equation becomes the equation of the graph that we drew:

$$y - (-5) = (x - 2)^2$$
$$y + 5 = x^2 - 4x + 4$$
$$y = x^2 - 4x - 1$$

The turning point of the parabola is the midpoint of the perpendicular segment from the fixed point to the fixed line. Since the coefficient of y in the equation of the parabola is 1, $4d = 1$ or $d = \frac{1}{4}$. The parabola opens upward so the fixed point is $\frac{1}{4}$ unit above the turning point and the fixed line is $\frac{1}{4}$ unit below the turning point. The coordinates of the fixed point are $\left(2, -4\frac{3}{4}\right)$ and the equation of the fixed line is $y = -5\frac{1}{4}$.

EXAMPLE 2

a. Draw the graph of $y = -x^2 - 2x + 8$ from $x = -4$ to $x = 2$.

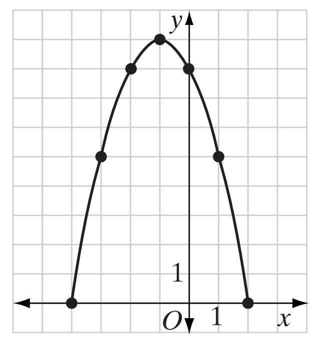
b. Write the coordinates of the turning point.

c. Write an equation of the axis of symmetry.

Solution **a.** (1) Make a table of values using integral values of x from $x = -4$ to $x = 2$.

(2) Plot the points whose coordinates are given in the table and draw a smooth curve through them.

x	$-x^2 - 2x + 8$	y
-4	$-16 + 8 + 8$	0
-3	$-9 + 6 + 8$	5
-2	$-4 + 4 + 8$	8
-1	$-1 + 2 + 8$	9
0	$0 - 0 + 8$	8
1	$-1 - 2 + 8$	5
2	$-4 - 4 + 8$	0



b. From the graph or from the table, the coordinates of the turning point appear to be $(-1, 9)$. We can verify this algebraically:

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1 \qquad y = -x^2 - 2x + 8 = -(-1)^2 - 2(-1) + 8 = 9$$

c. The axis of symmetry is the vertical line through the turning point, $x = -1$.

Here the parabola $y = x^2$ has been reflected in the x-axis so that the equation becomes $-y = x^2$. Then that parabola has been moved 1 unit to the left and 9 units up so that the equation becomes $-(y - 9) = (x - (-1))^2$ or $-y + 9 = x^2 + 2x + 1$, which can be written as $-y = x^2 + 2x - 8$ or $y = -x^2 - 2x + 8$.

EXAMPLE 3

A parabola is equidistant from a given point and a line. How does the turning point of the parabola relate to the given point and line?

Solution The x-coordinate of the turning point is the same as the x-coordinate of the given point and is halfway between the given point and the line.

EXAMPLE 4

Solve the following system of equations graphically and check:

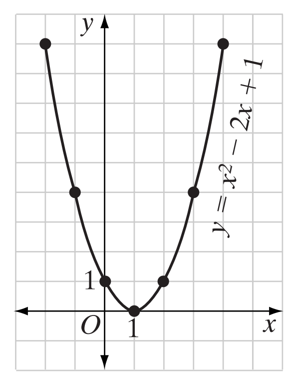
$$y = x^2 - 2x + 1$$
$$y = -x + 3$$

Solution (1) Make a table of values using at least three integral values of x that are less than that of the turning point and three that are greater. The x-coordinate of the turning point is:

$$\frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1$$

(2) Plot the points whose coordinates are given in the table and draw a smooth curve through them.

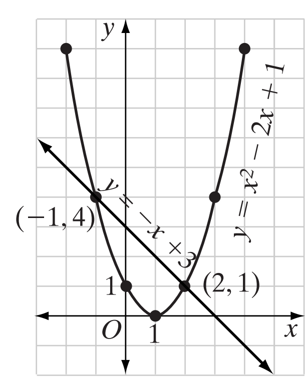
x	$x^2 - 2x + 1$	y
−2	4 + 4 + 1	9
−1	1 + 2 + 1	4
0	0 + 0 + 1	1
1	1 − 2 + 1	0
2	4 − 4 + 1	1
3	9 − 6 + 1	4
4	16 − 8 + 1	9



(3) On the same set of axes, sketch the graph of $y = -x + 3$.

The y-intercept is 3. Start at the point (0, 3).

The slope is −1 or $\frac{-1}{1}$. Move 1 unit down and 1 unit to the right to find a second point of the line. From this point, again move 1 unit down and 1 unit to the right to find a third point. Draw a line through these three points.



(4) Read the coordinates of the points of intersection from the graph. The common solutions are $(-1, 4)$ and $(2, 1)$.

Answer $(-1, 4)$ and $(2, 1)$ or $x = -1, y = 4$ and $x = 2, y = 1$

Exercises

Writing About Mathematics

1. Luis said that the solutions to the equation $-x^2 - 2x + 8 = 0$ are the x-coordinates of the points at which the graph of $y = -x^2 - 2x + 8$ intersects the y-axis. Do you agree with Luis? Explain why or why not.

2. Amanda said that if the turning point of a parabola is $(1, 0)$, then the x-axis is tangent to the parabola. Do you agree with Amanda? Explain why or why not.

Developing Skills

In 3–8, find the coordinates of the turning point and the equation of the axis of symmetry of each parabola.

3. $y = x^2 - 6x + 1$ **4.** $y = x^2 - 2x + 3$ **5.** $y = x^2 + 4x - 1$

6. $y = -x^2 + 2x + 5$ **7.** $y = -x^2 - 8x + 4$ **8.** $y = x^2 - 5x + 2$

In 9–16, find the common solution of each system of equations graphically and check your solution.

9. $y = x^2 - 2x + 2$
$y = x + 2$

10. $y = x^2 - 1$
$x + y = 1$

11. $y = x^2 - 4x + 3$
$y = x - 1$

12. $y = x^2 + 2x - 3$
$y = 1 - x$

13. $y = x^2 - 4x + 2$
$y = 2x - 3$

14. $y = -x^2 + 2x + 2$
$y = 2x - 2$

15. $y = -x^2 + 6x - 5$
$y = 7 - 2x$

16. $y = 2x - x^2$
$y = 2x - 4$

In 17–20 write the equation of the parabola that is the locus of points equidistant from each given point and line.

17. $F\left(0, \frac{1}{4}\right)$ and $y = -\frac{1}{4}$

18. $F\left(0, -\frac{1}{4}\right)$ and $y = \frac{1}{4}$

19. $F(0, 2)$ and $y = -2$

20. $F(3, -3)$ and $y = 3$

Hands-On Activity

If the graph of an equation is moved h units in the horizontal direction and k units in the vertical direction, then x is replaced by $x - h$ and y is replaced by $y - k$ in the given equation.

1. The turning point of the parabola $y = ax^2$ is $(0, 0)$. If the parabola $y = ax^2$ is moved so that the coordinates of the turning point are $(3, 5)$, what is the equation of the parabola?
2. If the parabola $y = ax^2$ is moved so that the coordinates of the turning point are (h, k), what is the equation of the parabola?

CHAPTER SUMMARY

Loci

- A **locus of points** is the set of all points, and only those points, that satisfy a given condition.
- The **locus of points equidistant from two fixed points** that are the endpoints of a segment is the perpendicular bisector of the segment.
- The **locus of points equidistant from two intersecting lines** is a pair of lines that bisect the angles formed by the intersecting lines.
- The **locus of points equidistant from two parallel lines** is a third line, parallel to the given lines and midway between them.
- The **locus of points at a fixed distance from a line** is a pair of lines, each parallel to the given line and at the fixed distance from the given line.
- The **locus of points at a fixed distance from a fixed point** is a circle whose center is the fixed point and whose radius is the fixed distance.
- The **locus of points equidistant from a fixed point and a line** is a parabola.

Loci in the Coordinate Plane

- The locus of points in the coordinate plane r units from (h, k) is the circle whose equation is $(x - h)^2 + (y - k)^2 = r^2$.
- The locus of points d units from the horizontal line $y = a$ is the pair of lines $y = a + d$ and $y = a - d$.
- The locus of points d units from the vertical line $x = a$ is the pair of lines $x = a + d$ and $x = a - d$.
- The locus of points equidistant from $A(a, c)$ and $B(b, d)$ is a line perpendicular to $\overline{AB}$ at its midpoint, $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$.
- The locus of points equidistant from the axes is the pair of lines $y = x$ and $y = -x$.
- The locus of points equidistant from the lines $y = mx$ and $y = -mx$ is the pair of lines $y = 0$ and $x = 0$, that is, the x-axis and the y-axis.
- The locus of points equidistant from $(h, k + d)$ and $y = k - d$ is the parabola whose equation is $4d(y - k) = (x - h)^2$.

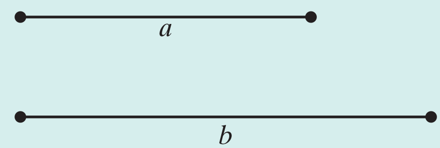
VOCABULARY

14-2 Locus • Concentric circles

14-3 Compound locus

REVIEW EXERCISES

1. *Construct*:



a. a right angle.

b. an angle whose measure is 45°.

c. parallelogram $ABCD$ with $AB = a$, $BC = b$ and $m\angle B = 45$.

2. Draw $\overline{PQ}$. Construct S on $\overline{PQ}$ such that $PS : SQ = 2 : 3$.

In 3–6, sketch and describe each locus.

3. Equidistant from two points that are 6 centimeters apart.

4. Four centimeters from A and equidistant from A and B, the endpoints of a line segment that is 6 centimeters long.

5. Equidistant from the endpoints of a line segment that is 6 centimeters long and 2 centimeters from the midpoint of the segment.

6. Equidistant from parallel lines that are 5 inches apart and 4 inches from a point on one of the given lines.

In 7–12, sketch the locus of points on graph paper and write the equation or equations of the locus.

7. 3 units from $(-1, 2)$.

8. 2 units from $(2, 4)$ and 2 units from $x = 2$.

9. Equidistant from $x = -1$ and $x = 5$.

10. Equidistant from $y = 2$ and $y = 8$.

11. Equidistant from $y = 2x$ and $y = -2x$.

12. Equidistant from $(1, -3)$ and $(3, 1)$.

13. Find the coordinates of the points on the line $x + y = 7$ that are 5 units from the origin.

14. **a.** Draw the graph of $y = x^2 - 4x - 1$.

b. On the same set of axes, draw the graph of $y = -2x + 2$.

c. What are the coordinates of the points of intersection of the graphs drawn in **a** and **b**?

In 15–18, solve each system of equations graphically.

15. $x^2 + y^2 = 25$
$y = x - 1$

16. $(x - 2)^2 + (y + 1)^2 = 4$
$x = 2$

17. $y = x^2 + 2x - 1$
$y = x + 1$

18. $y = -x^2 + 4x + 2$
$x + y = 6$

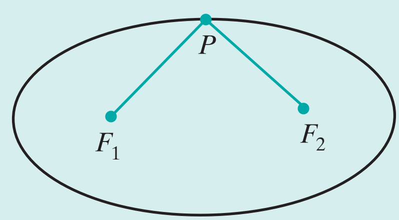
19. A field is rectangular in shape and measures 80 feet by 120 feet. How many points are equidistant from any three sides of the field? (*Hint:* Sketch the locus of points equidistant from each pair of sides of the field.)

20. The coordinates of the vertices of isosceles trapezoid $ABCD$ are $A(0, 0)$, $B(6, 0)$, $C(4, 4)$, and $D(2, 4)$. What are the coordinates of the locus of points equidistant from the vertices of the trapezoid? (*Hint:* Sketch the locus of points equidistant from each pair of vertices.)

Exploration

An **ellipse** is the locus of points such that the sum of the distances from two fixed points F_1 and F_2 is a constant, k. Use the following procedure to create an ellipse. You will need a piece of string, a piece of thick cardboard, and two thumbtacks.

STEP 1. Place two thumbtacks in the cardboard separated by a distance that is less than the length of the string. Call the thumbtacks F_1 and F_2. Attach one end of the string to F_1 and the other to F_2. The length of the string represents the sum of the distances from a point on the locus to the fixed points.



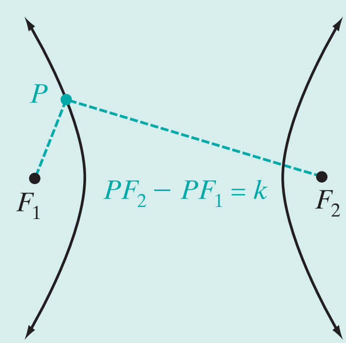
STEP 2. Place your pencil in the loop of string and pull the string taut to locate some point P. Keeping the string taut, slowly trace your pencil around the fixed points until you have created a closed figure.

a. Prove that the closed figure you created is an ellipse. (*Hint*: Let k be the length of the string.)

b. If F_1 and F_2 move closer and closer together, how is the shape of the ellipse affected?

A **hyperbola** is the locus of points such that the difference of the distances from fixed points F_1 and F_2 is a constant, k.

c. Explain why a hyperbola is not a closed figure.



CUMULATIVE REVIEW Chapters 1–14

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following has a midpoint?

(1) $\overleftrightarrow{AB}$ (2) $\overline{AB}$ (3) $\overrightarrow{AB}$ (4) $\angle ABC$

2. What is the inverse of the statement "If two lines segments are congruent, then they have equal measures?"

(1) If two line segments are not congruent, then they do not have equal measures.

(2) If two line segments do not have equal measures, then they are not congruent.

(3) If two line segments have equal measures, then they are congruent.

(4) Two line segments are not congruent if they have unequal measures.

3. In triangle ABC, $\overline{AD}$ is the altitude to $\overline{BC}$. If D is the midpoint of $\overline{BC}$, then which of the following may be false?

(1) $AB = AC$ (3) $AB = AD$

(2) $BD = CD$ (4) $\text{m}\angle B = \text{m}\angle C$

4. If two angles of a triangle are congruent and complementary, then the triangle is

(1) isosceles and right (3) isosceles and obtuse

(2) scalene and right (4) scalene and acute

5. The equation of a line that is perpendicular to the line $x + 3y = 6$ is

(1) $y = 3x + 1$ (3) $y = -\frac{1}{3}x + 1$

(2) $y = -3x + 1$ (4) $y = \frac{1}{3}x + 1$

6. Under a line reflection, which of the following is *not* preserved?

(1) angle measure (3) orientation

(2) collinearity (4) midpoint

7. Which of the following is *not* sufficient to prove that quadrilateral $ABCD$ is a parallelogram?

(1) $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$ (3) $\overline{AC}$ and $\overline{BD}$ bisect each other

(2) $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$ (4) $\overline{AC} \perp \overline{BD}$

8. A prism with bases that are equilateral triangles has a height of 12.0 centimeters. If the length of each side of a base is 8.00 centimeters, what is the number of square centimeters in the lateral area of the prism?

(1) 48.0 (2) 96.0 (3) 288 (4) 384

9. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of lengths 4 and 45. The length of the shorter leg is

(1) $6\sqrt{5}$ (2) 14 (3) $\sqrt{2{,}009}$ (4) $\sqrt{2{,}041}$

10. Triangle ABC is inscribed in circle O. If $m\widehat{AB} = 110$ and $m\widehat{BC} = 90$, what is the measure of $\angle ABC$?

(1) 45 (2) 55 (3) 80 (4) 110

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit

11. D is a point on side $\overline{AB}$ and E is a point on side $\overline{AC}$ of $\triangle ABC$ such that $\overline{DE} \parallel \overline{BC}$. If $AD = 6$, $DB = 9$, and $AC = 20$, find AE and EC.

12. A pile of gravel is in the form of a cone. The circumference of the pile of gravel is 75 feet and its height is 12 feet. How many cubic feet of gravel does the pile contain? Give your answer to the nearest hundred cubic feet.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. a. On graph paper, sketch the graph of $y = x^2 - x + 2$.

b. On the same set of axes, sketch the graph of $y = x + 5$.

c. What are the common solutions of the equations $y = x^2 - x + 2$ and $y = x + 5$?

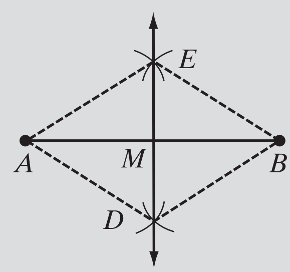
14. a. Show that the line whose equation is $x + y = 4$ intersects the circle whose equation is $x^2 + y^2 = 8$ in exactly one point and is therefore tangent to the circle.

b. Show that the radius to the point of tangency is perpendicular to the tangent.

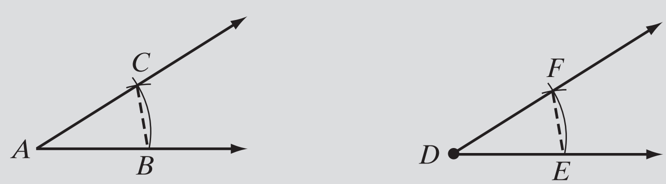
Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. a. The figure below shows the construction of the perpendicular bisector, $\overleftrightarrow{DE}$, of segment $\overline{AB}$. Identify all congruent segments and angles in the construction and state the theorems that prove that $\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AB}$.



b. The figure below shows the construction of $\angle FDE$ congruent to $\angle CAB$. Identify all congruent lines and angles in the construction and state the theorems that prove that $\angle FDE \cong \angle CAB$.



16. a. Quadrilateral $ABCD$ is inscribed in circle O, $\overline{AB} \parallel \overline{CD}$, and $\overline{AC}$ is a diameter. Prove that $ABCD$ is a rectangle.

b. In quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. If $\triangle ABE$ and $\triangle CDE$ are similar but not congruent, prove that the quadrilateral is a trapezoid.

c. Triangle ABC is equilateral. From D, the midpoint of $\overline{AB}$, a line segment is drawn to E, the midpoint of $\overline{BC}$, and to F, the midpoint of $\overline{AC}$. Prove that $DECF$ is a rhombus.