CHAPTER

8

SLOPES AND EQUATIONS OF LINES

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In coordinate geometry, a straight line can be characterized as a line whose slope is a constant. Do curves have slopes and if so, can they be determined?

In the late 17th and early 18th centuries, two men independently developed methods to answer these and other questions about curves and the areas that they determine. The slope of a curve at a point is defined to be the slope of the *tangent* to that curve at that point. Descartes worked on the problem of finding the slope of a tangent to a curve by considering the slope of a tangent to a circle that intersected the curve at a given point and had its center on an axis. Gottfried Leibniz (1646–1716) in Germany and Isaac Newton (1642–1727) in England each developed methods for determining the slope of a tangent to a curve at any point as well as methods for determining the area bounded by a curve or curves. Newton acknowledged the influence of the work of mathematicians and scientists who preceded him in his statement, "If I have seen further, it is by standing on the shoulders of giants." The work of Leibniz and Newton was the basis of differential and integral calculus.

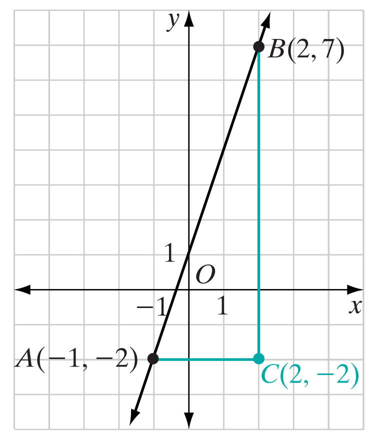
8-1 THE SLOPE OF A LINE

In the coordinate plane, horizontal and vertical lines are used as reference lines. Slant lines intersect horizontal lines at acute and obtuse angles. The ratio that measures the slant of a line in the coordinate plane is the *slope* of the line.

Finding the Slope of a Line

Through two points, one and only one line can be drawn. In the coordinate plane, if the coordinates of two points are given, it is possible to use a ratio to determine the measure of the slant of the line. This ratio is the **slope** of the line.

Through the points $A(-1, -2)$ and $B(2, 7)$, $\overleftrightarrow{AB}$ is drawn. Let $C(2, -2)$ be the point at which the vertical line through B intersects the horizontal line through A. The slope of $\overline{AB}$ is the ratio of the change in vertical distance, BC, to the change in horizontal distance, AC. Since B and C are on the same vertical line, BC is the difference in the y-coordinates of B and C. Since A and C are on the same horizontal line, AC is the difference in the x-coordinates of A and C.



$$\begin{aligned}\text{slope of } \overline{AB} &= \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}\\ &= \frac{BC}{AC}\\ &= \frac{7 - (-2)}{2 - (-1)}\\ &= \frac{9}{3}\\ &= 3\end{aligned}$$

This ratio is the same for any segment of the line $\overleftrightarrow{AB}$. Suppose we change the order of the points $(-1, -2)$ and $(2, 7)$ in performing the computation. We then have:

$$\begin{aligned}\text{slope of } \overline{BA} &= \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}\\ &= \frac{CB}{CA}\\ &= \frac{(-2) - 7}{(-1) - 2}\\ &= \frac{-9}{-3}\\ &= 3\end{aligned}$$

The result of both computations is the same. When we compute the slope of a line that is determined by two points, it does not matter which point is considered as the first point and which the second.

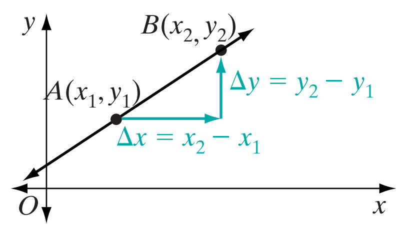
Also, when we find the slope of a line using two points on the line, it does not matter which two points on the line we use because all segments of a line have the same slope as the line.

Procedure

To find the slope of a line:

1. Select any two points on the line.
2. Find the vertical change, that is, the change in *y*-values by subtracting the *y*-coordinates in any order.
3. Find the horizontal change, that is, the change in *x*-values, by subtracting the *x*-coordinates in the same order as the *y*-coordinates.
4. Write the ratio of the vertical change to the horizontal change.

In general, the slope, m, of a line that passes through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 \neq x_2$, is the ratio of the difference of the y-values of these points to the difference of the corresponding x-values.



$$\text{slope of } \overleftrightarrow{AB} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

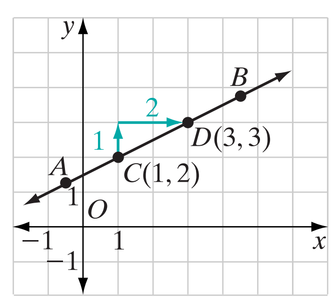
The difference in x-values, $x_2 - x_1$, can be represented by $\boldsymbol{\Delta x}$, read as "delta x." Similarly, the difference in y-values, $y_2 - y_1$, can be represented by $\boldsymbol{\Delta y}$, read as "delta y." Therefore, we write:

$$\textbf{slope of a line} = m = \frac{\Delta y}{\Delta x}$$

The slope of a line is positive if the line slants upward from left to right, negative if the line slants downward from left to right, or zero if the line is horizontal. If the line is vertical, it has no slope.

Positive Slope

The points C and D are two points on $\overleftrightarrow{AB}$. Let the coordinates of C be (1, 2) and the coordinates of D be (3, 3). As the values of x increase, the values of y also increase. The graph of $\overleftrightarrow{AB}$ slants upward.

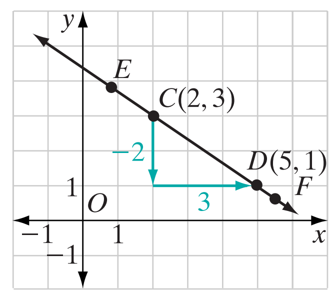


$$\text{slope of } \overleftrightarrow{AB} = \frac{3 - 2}{3 - 1} = \frac{1}{2}$$

The slope of $\overleftrightarrow{AB}$ is positive.

Negative Slope

Points C and D are two points on $\overleftrightarrow{EF}$. Let the coordinates of C be (2, 3) and the coordinates of D be (5, 1). As the values of x increase, the values of y decrease. The graph of $\overleftrightarrow{EF}$ slants downward.

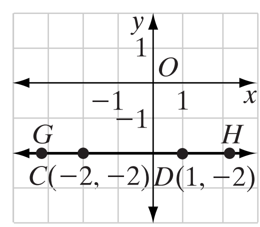


$$\text{slope of } \overleftrightarrow{EF} = \frac{1-3}{5-2} = -\frac{2}{3}$$

The slope of $\overleftrightarrow{EF}$ is negative.

Zero Slope

Points C and D are two points on $\overleftrightarrow{GH}$. Let the coordinates of C be (−2, −2) and the coordinates of D be (1, −2). As the values of x increase, the values of y remain the same. The graph of $\overleftrightarrow{GH}$ is a horizontal line.

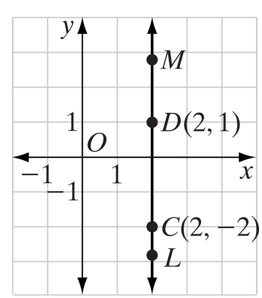


$$\text{slope of } \overleftrightarrow{GH} = \frac{-2-(-2)}{1-(-2)} = \frac{0}{3} = 0$$

The slope of $\overleftrightarrow{GH}$ is 0.
The slope of any horizontal line is 0.

No Slope

Points C and D are two points on $\overleftrightarrow{ML}$. Let the coordinates of C be (2, −2) and the coordinates of D be (2, 1). The values of x remain the same for all points as the values of y increase. The graph of $\overleftrightarrow{ML}$ is a vertical line.



The slope of $\overleftrightarrow{ML}$ is $\frac{-2-1}{2-2} = \frac{-3}{0}$, which is undefined.

$\overleftrightarrow{ML}$ has no slope.
A vertical line has no slope.

A fundamental property of a straight line is that its slope is constant. Therefore, any two points on a line may be used to compute the slope of the line.

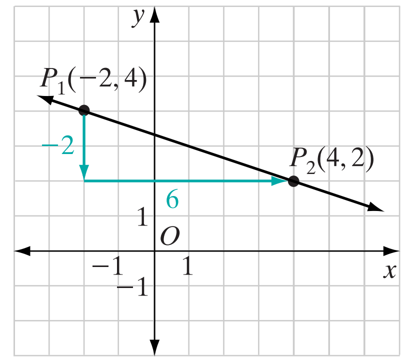
EXAMPLE 1

Find the slope of the line that is determined by points $(-2, 4)$ and $(4, 2)$.

Solution Plot points $(-2, 4)$ and $(4, 2)$.

Let point $(-2, 4)$ be $P_1(x_1, y_1)$ and let point $(4, 2)$ be $P_2(x_2, y_2)$.
Then, $x_1 = -2$, $y_1 = 4$, $x_2 = 4$, and $y_2 = 2$.

$$\text{slope of } \overleftrightarrow{P_1P_2} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 4}{4 - (-2)} = \frac{-2}{6} = -\frac{1}{3} \quad \textit{Answer}$$



EXAMPLE 2

Through point $(1, 4)$, draw the line whose slope is $-\frac{3}{2}$.

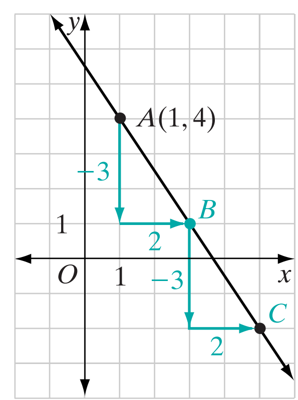
Solution *How to Proceed*

(1) Graph point $A(1, 4)$.

(2) Note that, since slope $= \frac{\Delta y}{\Delta x} = -\frac{3}{2} = \frac{-3}{2}$, when y decreases by 3, x increases by 2. Start at point $A(1, 4)$ and move 3 units downward and 2 units to the right to locate point B.

(3) Start at B and repeat these movements to locate point C.

(4) Draw a line that passes through points A, B, and C.



Exercises

Writing About Mathematics

1. How is the symbol Δy read and what is its meaning?

2. Brad said that since 0 is the symbol for "nothing," no slope is the same as zero slope. Do you agree with Brad? Explain why or why not.

Developing Skills

In 3–11, in each case: **a.** Plot both points and draw the line that they determine. **b.** Find the slope of this line if the line has a defined slope. **c.** State whether the line through these points would slant upward, slant downward, be horizontal, or be vertical.

3. $(0, 1)$ and $(4, 5)$
4. $(1, 0)$ and $(4, 9)$
5. $(0, 0)$ and $(-3, 6)$
6. $(-1, 5)$ and $(3, 9)$
7. $(5, -3)$ and $(1, -1)$
8. $(-2, 4)$ and $(-2, 2)$
9. $(-1, -2)$ and $(7, -8)$
10. $(4, 2)$ and $(8, 2)$
11. $(-1, 3)$ and $(2, -3)$

In 12–23, in each case, draw a line with the given slope, *m*, through the given point.

12. $(0, 1); m = 2$
13. $(-1, 3); m = 3$
14. $(2, 5); m = -1$
15. $(-4, 5); m = \frac{2}{3}$
16. $(-3, 2); m = 0$
17. $(-4, 7); m = -2$
18. $(1, 3); m = 1$
19. $(-2, 3); m = -\frac{3}{2}$
20. $(-1, 5); m = -\frac{1}{3}$
21. $(-1, 0); m = \frac{5}{4}$
22. $(0, -2); m = \frac{2}{3}$
23. $(-2, 0); m = \frac{1}{2}$

Applying Skills

24. a. Graph the points $A(2, 4)$ and $B(8, 4)$.
b. From point *A*, draw a line that has a slope of 2.
c. From point *B*, draw a line that has a slope of -2.
d. Let the intersection of the lines drawn in **b** and **c** be *C*. What are the coordinates of *C*?
e. Draw the altitude from vertex *C* to base $\overline{AB}$ of $\triangle ABC$. Prove that $\triangle ABC$ is an isosceles triangle.

25. Points $A(3, -2)$ and $B(9, -2)$ are two vertices of rectangle *ABCD* whose area is 24 square units. Find the coordinates of *C* and *D*. (Two answers are possible.)

26. A path to the top of a hill rises 75 feet vertically in a horizontal distance of 100 feet. Find the slope of the path up the hill.

27. The doorway of a building under construction is 3 feet above the ground. A ramp to reach the doorway is to have a slope of $\frac{2}{5}$. How far from the base of the building should the ramp begin?

8-2 THE EQUATION OF A LINE

We have learned two facts that we can use to write the equation of a line.

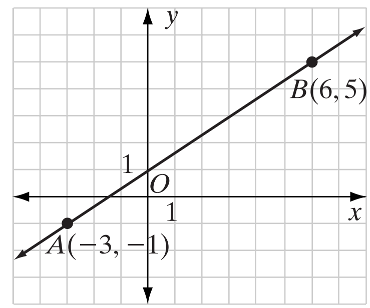
- ► **Two points determine a line.**
- ► **The slope of a straight line is constant.**

The second statement on the bottom of page 295 can be written as a biconditional:

Postulate 8.1

> A, B, and C lie on the same line if and only if the slope of $\overline{AB}$ is equal to the slope of $\overline{BC}$.

Let $A(-3, -1)$ and $B(6, 5)$ be two points on $\overleftrightarrow{AB}$. Let $P(x, y)$ be any other point on $\overleftrightarrow{AB}$. We can write the equation of $\overleftrightarrow{AB}$ by using the following fact:



$$\begin{aligned}
\text{slope of } \overline{AB} &= \text{slope of } \overline{BP} \\
\frac{-1-5}{-3-6} &= \frac{y-5}{x-6} \\
\frac{-6}{-9} &= \frac{y-5}{x-6} \\
\frac{2}{3} &= \frac{y-5}{x-6} \\
3(y-5) &= 2(x-6) \\
3y-15 &= 2x-12 \\
3y &= 2x+3 \\
y &= \tfrac{2}{3}x+1
\end{aligned}$$

Recall that when the equation is solved for y in terms of x, the coefficient of x is the slope of the line and the constant term is the ***y*-intercept**, the y-coordinate of the point where the line intersects the y-axis.

The ***x*-intercept** is the x-coordinate of the point where the line intersects the x-axis.

Procedure

To find the equation of a line given two points on the line:

1. Find the slope of the line using the coordinates of the two given points.
2. Let $P(x, y)$ be any point on the line. Write a ratio that expresses the slope of the line in terms of the coordinates of P and the coordinates of one of the given points.
3. Let the slope found in step 2 be equal to the slope found in step 1.
4. Solve the equation written in step 3 for y.

When we are given the coordinates of one point and the slope of the line, the equation of the line can be determined. For example, if (a, b) is a point on the line whose slope is m, then the equation is:

$$\frac{y - b}{x - a} = m$$

This equation is called the **point-slope form** of the equation of a line.

EXAMPLE 1

The slope of a line through the point $A(3, 0)$ is -2.

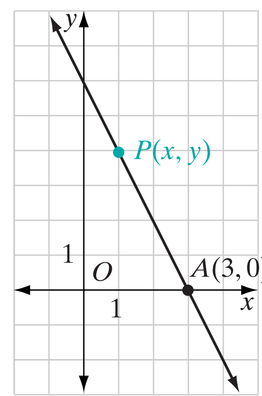
a. Use the point-slope form to write an equation of the line.

b. What is the y-intercept of the line?

c. What is the x-intercept of the line?

Solution **a.** Let $P(x, y)$ be any point on the line. The slope of $\overline{AP} = -2$.

$$\frac{y - 0}{x - 3} = -2$$
$$y = -2(x - 3)$$
$$y = -2x + 6$$



b. The y-intercept is the y-coordinate of the point at which the line intersects the y-axis, that is, the value of y when x is 0. When $x = 0$,

$$y = -2(0) + 6$$
$$= 0 + 6$$
$$= 6$$

The y-intercept is 6. When the equation is solved for y, the y-intercept is the constant term.

c. The x-intercept is the x-coordinate of the point at which the line intersects the x-axis, that is, the value of x when y is 0. Since $(3, 0)$ is a given point on the line, the x-intercept is 3.

Answers **a.** $y = -2x + 6$ **b.** 6 **c.** 3

EXAMPLE 2

a. Show that the three points $A(-2, -3)$, $B(0, 1)$, and $C(3, 7)$ lie on a line.

b. Write an equation of the line through A, B, and C.

Solution **a.** The points $A(-2, -3)$, $B(0, 1)$, and $C(3, 7)$ lie on the same line if and only if the slope of $\overline{AB}$ is equal to the slope of $\overline{BC}$.

$$\text{slope of } \overline{AB} = \frac{-3 - 1}{-2 - 0} = \frac{-4}{-2} = 2 \qquad \text{slope of } \overline{BC} = \frac{1 - 7}{0 - 3} = \frac{-6}{-3} = 2$$

The slopes are equal. Therefore, the three points lie on the same line.

b. Use the point-slope form of the equation of a line. Let (x, y) be any other point on the line. You can use any of the points, A, B, or C, with (x, y) and the slope of the line, to write an equation. We will use $A(-2, -3)$.

$$\frac{y - (-3)}{x - (-2)} = 2$$
$$y + 3 = 2(x + 2)$$
$$y + 3 = 2x + 4$$
$$y = 2x + 1$$

Answers **a.** Since the slope of $\overline{AB}$ is equal to the slope of $\overline{BC}$, A, B, and C lie on a line.

b. $y = 2x + 1$

Alternative Solution

(1) Write the **slope-intercept form** of an equation of a line: $y = mx + b$

(2) Substitute the coordinates of A in that equation: $-3 = m(-2) + b$

(3) Substitute the coordinates of C in that equation: $7 = m(3) + b$

(4) Write the system of two equations from (2) and (3):
$$-3 = -2m + b$$
$$7 = 3m + b$$

(5) Solve the equation $-3 = -2m + b$ for b in terms of m: $b = 2m - 3$

(6) Substitute the value of b found in (5) for b in the second equation and solve for m:
$$7 = 3m + b$$
$$7 = 3m + (2m - 3)$$
$$7 = 5m - 3$$
$$10 = 5m$$
$$2 = m$$

(7) Substitute this value of m in either equation to find the value of b:
$$b = 2m - 3$$
$$b = 2(2) - 3$$
$$b = 1$$

The equation is $y = 2x + 1$.

We can show that each of the given points lies on the line whose equation is $y = 2x + 1$ by showing that each pair of values makes the equation true.

$(-2, -3)$	$(0, 1)$	$(3, 7)$
$y = 2x + 1$	$y = 2x + 1$	$y = 2x + 1$
$-3 \stackrel{?}{=} 2(-2) + 1$	$1 \stackrel{?}{=} 2(0) + 1$	$7 \stackrel{?}{=} 2(3) + 1$
$-3 = -3$ ✔	$1 = 1$ ✔	$7 = 7$ ✔

Answers **a, b:** Since the coordinates of each point make the equation $y = 2x + 1$ true, the three points lie on a line whose equation is $y = 2x + 1$.

Exercises

Writing About Mathematics

1. Jonah said that A, B, C, and D lie on the same line if the slope of $\overline{AB}$ is equal to the slope of $\overline{CD}$. Do you agree with Jonah? Explain why or why not.

2. Sandi said that the point-slope form cannot be used to find the equation of a line with no slope.

a. Do you agree with Sandi? Justify your answer.

b. Explain how you can find the equation of a line with no slope.

Developing Skills

In 3–14, write the equation of each line.

3. Through $(1, -2)$ and $(5, 10)$

4. Through $(0, -1)$ and $(1, 0)$

5. Through $(2, -2)$ and $(0, 6)$

6. Slope 2 and through $(-2, -4)$

7. Slope -4 through $(1, 1)$

8. Slope $\frac{1}{2}$ through $(5, 4)$

9. Slope -3 and y-intercept 5

10. Slope 1 and x-intercept 4

11. Through $(1, 5)$ and $(4, 5)$

12. Through $(1, 5)$ and $(1, -2)$

13. x-intercept 2 and y-intercept 4

14. No slope and x-intercept 2

15. a. Do the points $P(3, 3)$, $Q(5, 4)$, and $R(-1, 1)$ lie on the same line?

b. If P, Q, and R lie on the same line, find the equation of the line. If P, Q, and R do not lie on the same line, find the equations of the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{PR}$.

16. a. Do the points $L(1, 3)$, $M(5, 6)$, and $N(-4, 0)$ lie on the same line?

b. If L, M, and N lie on the same line, find the equation of the line. If L, M, and N do not lie on the same line, find the equations of the lines $\overleftrightarrow{LM}$, $\overleftrightarrow{MN}$, and $\overleftrightarrow{LN}$.

Applying Skills

17. At a TV repair shop, there is a uniform charge for any TV brought in for repair plus an hourly fee for the work done. For a TV that needed two hours of repair, the cost was $100. For a TV that needed one and a half hours of repair, the cost was $80.

a. Write an equation that can be used to find the cost of repair, y, when x hours of work are required. Write the given information as ordered pairs, (2, 100) and (1.5, 80).

b. What would be the cost of repairing a TV that requires 3 hours of work?

c. What does the coefficient of x in the equation that you wrote in **a** represent?

d. What does the constant term in the equation that you wrote in **a** represent?

18. An office manager buys printer cartridges from a mail order firm. The bill for each order includes the cost of the cartridges plus a shipping cost that is the same for each order. The bill for 5 cartridges was $98 and a later bill for 3 cartridges, at the same rate, was $62.

a. Write an equation that can be used to find y, the amount of the bill for an order, when x cartridges are ordered. Write the given information as ordered pairs, (5, 98) and (3, 62).

b. What would be the amount of the bill for 8 cartridges?

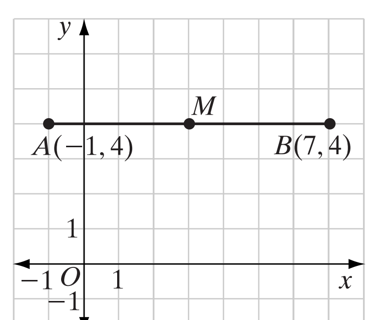
c. What does the coefficient of x in the equation that you wrote in **a** represent?

d. What does the constant term in the equation that you wrote in **a** represent?

19. Show that if the equation of the line can be written as $\frac{x}{a} + \frac{y}{b} = 1$, then the line intersects the x-axis at $(a, 0)$ and the y-axis at $(0, b)$.

8-3 MIDPOINT OF A LINE SEGMENT

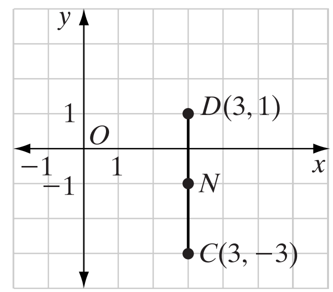
The *midpoint* of a line segment is the point of that line segment that divides the segment into two congruent segments. In the figure, $A(-1, 4)$ and $B(7, 4)$ determine a horizontal segment, $\overline{AB}$, whose midpoint, M, can be found by first finding the distance from A to B. Since $AB = 7 - (-1) = 8$ units, $AM = 4$ units, and $MB = 4$ units.



We can find the x-coordinate of M by adding AM to the x-coordinate of A or by subtracting MB from the x-coordinate of B. The x-coordinate of M is $-1 + 4 = 3$ or $7 - 4 = 3$. Since A, B, and M are all on the same horizontal line, they all have the same y-coordinate, 4. The coordinates of the midpoint M are (3, 4). The x-coordinate of M is the average of the x-coordinates of A and B.

$$\begin{aligned} x\text{-coordinate of } M &= \frac{-1 + 7}{2} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

Similarly, $C(3, -3)$ and $D(3, 1)$ determine a vertical segment, $\overline{CD}$, whose midpoint, N, can be found by first finding the distance from C to D. Since $CD = 1 - (-3) = 4$ units, $CN = 2$ units, and $ND = 2$ units. We can find the y-coordinate of N by adding 2 to the y-coordinate of C or by subtracting 2 from the y-coordinate of D. The y-coordinate of N is $-3 + 2 = -1$ or $1 - 2 = -1$. Since C, D, and N are all on the same vertical line, they all have the same x-coordinate, 3. The coordinates of the midpoint N are $(3, -1)$. The y-coordinate of N is the average of the y-coordinates of C and D.



$$\begin{aligned} y\text{-coordinate of } N &= \frac{1 + (-3)}{2} \\ &= \frac{-2}{2} \\ &= -1 \end{aligned}$$

These examples suggest the following relationships:

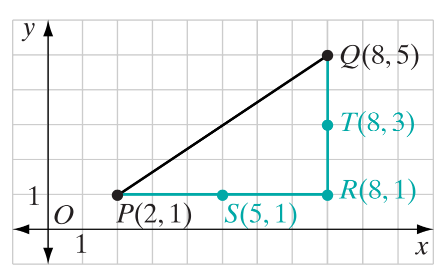
- **If the endpoints of a horizontal segment are (a, c) and (b, c), then the coordinates of the midpoint are:**

$$\left(\frac{a + b}{2}, c\right)$$

- **If the endpoints of a vertical segment are (d, e) and (d, f), then the coordinates of the midpoint are:**

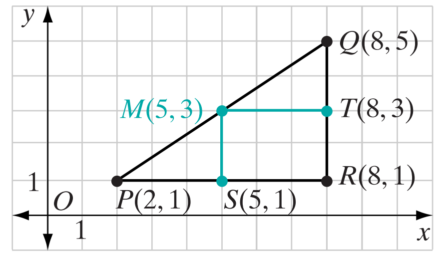
$$\left(d, \frac{e + f}{2}\right)$$

In the figure, $P(2, 1)$ and $Q(8, 5)$ are the endpoints of $\overline{PQ}$. A horizontal line through P and a vertical line through Q intersect at $R(8, 1)$.



- The coordinates of S, the midpoint of $\overline{PR}$, are $\left(\frac{2 + 8}{2}, 1\right) = (5, 1)$.
- The coordinates of T, the midpoint of $\overline{QR}$, are $\left(8, \frac{5 + 1}{2}\right) = (8, 3)$.

Now draw a vertical line through S and a horizontal line through T. These lines appear to intersect at a point on $\overline{PQ}$ that we will call M. This point has the coordinates (5, 3). We need to show that this point is a point on $\overline{PQ}$ and is the midpoint of $\overline{PQ}$.



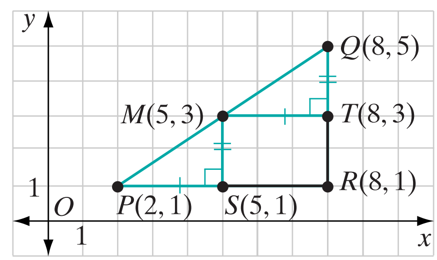
The point M is on $\overline{PQ}$ if and only if the slope of $\overline{PM}$ is equal to the slope of $\overline{MQ}$.

$$\text{slope of } \overline{PM} = \frac{3-1}{5-2} = \frac{2}{3} \qquad \text{slope of } \overline{MQ} = \frac{5-3}{8-5} = \frac{2}{3}$$

Since these slopes are equal, P, M, and Q lie on a line.

The point M is the midpoint of $\overline{PQ}$ if $PM = MQ$. We can show that $PM = MQ$ by showing that they are corresponding parts of congruent triangles.

- $PS = 5 - 2 = 3$ and $MT = 8 - 5 = 3$. Therefore, $\overline{PS} \cong \overline{MT}$.
- $SM = 3 - 1 = 2$ and $TQ = 5 - 3 = 2$. Therefore, $\overline{SM} \cong \overline{TQ}$.
- Since vertical lines are perpendicular to horizontal lines, $\angle PSM$ and $\angle MTQ$ are right angles and therefore congruent.



- Therefore, $\triangle PSM \cong \triangle MTQ$ by SAS and $\overline{PM} \cong \overline{MQ}$ because corresponding parts of congruent triangles are congruent.

We can conclude that the coordinates of the midpoint of a line segment whose endpoints are (2, 1) and (8, 5) are $\left(\frac{2+8}{2}, \frac{1+5}{2}\right) = (5, 3)$.

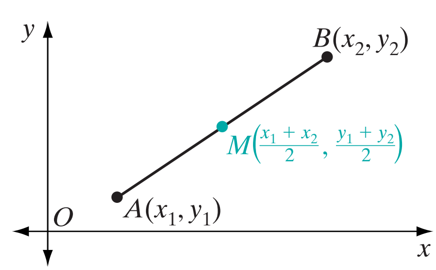
This example suggests the following theorem:

Theorem 8.1

> If the endpoints of a line segment are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint of the segment are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Given The endpoints of $\overline{AB}$ are $A(x_1, y_1)$ and $B(x_2, y_2)$.

Prove The coordinates of the midpoint of $\overline{AB}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



Proof In this proof we will use the following facts from previous chapters that we have shown to be true:

- Three points lie on the same line if the slope of the segment joining two of the points is equal to the slope of the segment joining one of these points to the third.
- If two points lie on the same horizontal line, they have the same y-coordinate and the length of the segment joining them is the absolute value of the difference of their x-coordinates.
- If two points lie on the same vertical line, they have the same x-coordinate and the length of the segment joining them is the absolute value of the difference of their y-coordinates.

We will follow a strategy similar to the one used in the previous example. First, we will prove that the point M with coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is on $\overline{AB}$, and then we will use congruent triangles to show that $\overline{AM} \cong \overline{MB}$. From the definition of a midpoint of a segment, this will prove that M is the midpoint of $\overline{AB}$.

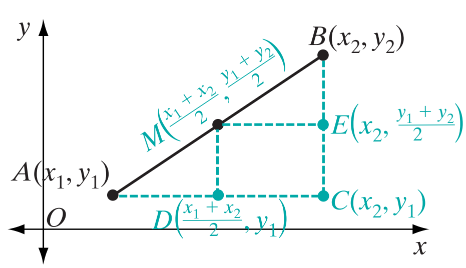
(1) Show that $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ lies on $\overline{AB}$:

$$\text{slope of } \overline{AM} = \frac{\frac{y_1 + y_2}{2} - y_1}{\frac{x_1 + x_2}{2} - x_1} = \frac{y_1 + y_2 - 2y_1}{x_1 + x_2 - 2x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope of } \overline{MB} = \frac{y_2 - \frac{y_1 + y_2}{2}}{x_2 - \frac{x_1 + x_2}{2}} = \frac{2y_2 - (y_1 + y_2)}{2x_2 - (x_1 + x_2)} = \frac{y_2 - y_1}{x_2 - x_1}$$

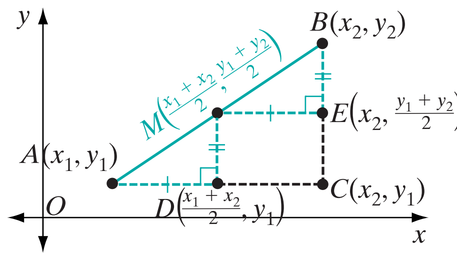
Points A, M, and B lie on the same line because the slope of $\overline{AM}$ is equal to the slope of $\overline{MB}$.

(2) Let C be the point on the same vertical line as B and the same horizontal line as A. The coordinates of C are (x_2, y_1).



The midpoint of $\overline{AC}$ is $D\left(\frac{x_1 + x_2}{2}, y_1\right)$.

The midpoint of $\overline{BC}$ is $E\left(x_2, \frac{y_1 + y_2}{2}\right)$.



$$AD = \left|\frac{x_1 + x_2}{2} - x_1\right| = \left|\frac{x_1 + x_2 - 2x_1}{2}\right| = \left|\frac{x_2 - x_1}{2}\right|$$

$$ME = \left|x_2 - \frac{x_1 + x_2}{2}\right| = \left|\frac{2x_2 - x_1 - x_2}{2}\right| = \left|\frac{x_2 - x_1}{2}\right|$$

Therefore, $AD = ME$ and $\overline{AD} \cong \overline{ME}$.

$$MD = \left|y_1 - \frac{y_1 + y_2}{2}\right| = \left|\frac{2y_1 - y_1 - y_2}{2}\right| = \left|\frac{y_1 - y_2}{2}\right|$$

$$BE = \left|\frac{y_1 + y_2}{2} - y_2\right| = \left|\frac{y_1 + y_2 - 2y_2}{2}\right| = \left|\frac{y_1 - y_2}{2}\right|$$

Therefore, $MD = BE$ and $\overline{MD} \cong \overline{BE}$.

Vertical lines are perpendicular to horizontal lines. $\overline{AD} \perp \overline{MD}$ and $\overline{ME} \perp \overline{BE}$. Therefore, $\angle ADM$ and $\angle MEB$ are right angles and are congruent.

$\triangle ADM \cong \triangle MEB$ by SAS and $\overline{AM} \cong \overline{MB}$. Therefore, $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is the midpoint of $\overline{AB}$.

We generally refer to the formula given in this theorem, that is, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, as the **midpoint formula**.

EXAMPLE 1

Find the coordinates of the midpoint of the segment $\overline{CD}$ whose endpoints are $C(-1, 5)$ and $D(4, -1)$.

Solution Let $(x_1, y_1) = (-1, 5)$ and $(x_2, y_2) = (4, -1)$.

The coordinates of the midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-1 + 4}{2}, \frac{5 + (-1)}{2}\right) = \left(\frac{3}{2}, \frac{4}{2}\right) = \left(\frac{3}{2}, 2\right)$$ *Answer*

EXAMPLE 2

$M(1, -2)$ is the midpoint of $\overline{AB}$ and the coordinates of A are $(-3, 2)$. Find the coordinates of B.

Solution Let the coordinates of $A = (x_1, y_1) = (-3, 2)$ and the coordinates of $B = (x_2, y_2)$.

The coordinates of the midpoint are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = (1, -2)$.

$$\frac{-3 + x_2}{2} = 1 \qquad\qquad \frac{2 + y_2}{2} = -2$$
$$-3 + x_2 = 2 \qquad\qquad 2 + y_2 = -4$$
$$x_2 = 5 \qquad\qquad y_2 = -6$$

Answer The coordinates of B are $(5, -6)$.

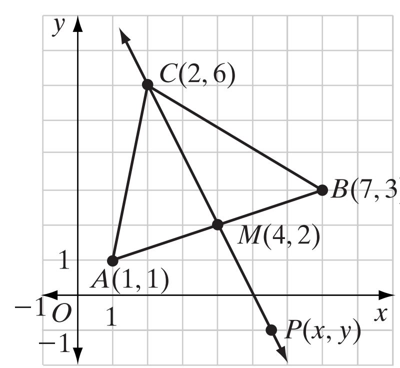
EXAMPLE 3

The vertices of $\triangle ABC$ are $A(1, 1)$, $B(7, 3)$, and $C(2, 6)$. Write an equation of the line that contains the median from C to $\overline{AB}$.

Solution A median of a triangle is a line segment that joins any vertex to the midpoint of the opposite side. Let M be the midpoint of $\overline{AB}$.

(1) Find the coordinates of M. Let (x_1, y_1) be $(1, 1)$ and (x_2, y_2) be $(7, 3)$. The coordinates of M are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + 7}{2}, \frac{1 + 3}{2}\right)$$
$$= (4, 2)$$



(2) Write the equation of the line through $C(2, 6)$ and $M(4, 2)$. Let $P(x, y)$ be any other point on the line.

$$\text{slope of } \overline{PC} = \text{slope of } \overline{CM}$$
$$\frac{y - 6}{x - 2} = \frac{6 - 2}{2 - 4}$$
$$\frac{y - 6}{x - 2} = -2$$
$$y - 6 = -2(x - 2)$$
$$y - 6 = -2x + 4$$
$$y = -2x + 10$$

Answer $y = -2x + 10$

Exercises

Writing About Mathematics

1. If $P(a, c)$ and $Q(b, c)$ are two points in the coordinate plane, show that the coordinates of the midpoint are $\left(a + \frac{b-a}{2}, c\right)$. $\left(\textit{Hint:} \text{ Show that } a + \frac{b-a}{2} = \frac{a+b}{2}.\right)$

2. If $P(a, c)$ and $Q(b, c)$ are two points in the coordinate plane, show that the coordinates of the midpoint are $\left(b - \frac{b-a}{2}, c\right)$. $\left(\textit{Hint:} \text{ Show that } b - \frac{b-a}{2} = \frac{a+b}{2}.\right)$

Developing Skills

In 3–14, find the midpoint of the each segment with the given endpoints.

3. $(1, 7), (5, 1)$
4. $(-2, 5), (8, 7)$
5. $(0, 8), (10, 0)$
6. $(0, -2), (4, 6)$
7. $(-5, 1), (5, -1)$
8. $(6, 6), (2, 5)$
9. $(1, 0), (0, 8)$
10. $(-3, 8), (5, 8)$
11. $(-3, -5), (-1, -1)$
12. $(7, -2), (-1, 9)$
13. $\left(\frac{1}{2}, 3\right), \left(1, 2\frac{1}{2}\right)$
14. $\left(\frac{1}{3}, 9\right), \left(\frac{2}{3}, 3\right)$

In 15–20, M is the midpoint of $\overline{AB}$. Find the coordinates of the third point when the coordinates of two of the points are given.

15. $A(2, 7), M(1, 6)$
16. $A(3, 3), M(3, 9)$
17. $B(4, 7), M(5, 5)$
18. $B(4, -2), M\left(\frac{3}{2}, 0\right)$
19. $A(3, 3), B(1, 10)$
20. $A(0, 7), M\left(0, \frac{7}{2}\right)$

Applying Skills

21. The points $A(1, 1)$ and $C(9, 7)$ are the vertices of rectangle $ABCD$ and B is a point on the same horizontal line as A.

a. What are the coordinates of vertices B and D?

b. Show that the midpoint of diagonal $\overline{AC}$ is also the midpoint of diagonal $\overline{BD}$.

22. The points $P(x_1, y_1)$ and $R(x_2, y_2)$ are vertices of rectangle $PQRS$ and Q is a point on the same horizontal line as P.

a. What are the coordinates of vertices Q and S?

b. Show that the midpoint of diagonal $\overline{PR}$ is also the midpoint of diagonal $\overline{QS}$.

23. The vertices of $\triangle ABC$ are $A(-1, 4)$, $B(5, 2)$, and $C(5, 6)$.

a. What are the coordinates of M, the midpoint of $\overline{AB}$?

b. Write an equation of $\overleftrightarrow{CM}$ that contains the median from C.

c. What are the coordinates of N, the midpoint of $\overline{AC}$?

d. Write an equation of $\overleftrightarrow{BN}$ that contains the median from B.

e. What are the coordinates of the intersection of $\overleftrightarrow{CM}$ and $\overleftrightarrow{BN}$?

f. What are the coordinates of P, the midpoint of $\overline{BC}$?

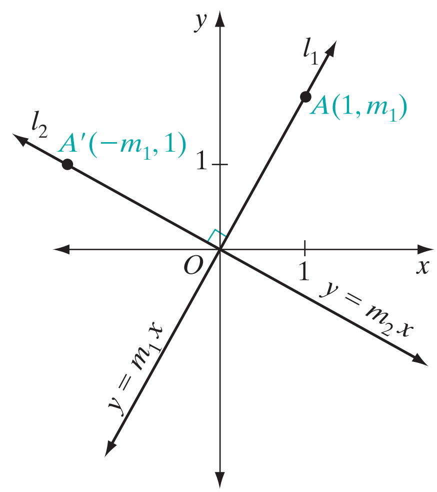
g. Write an equation of $\overleftrightarrow{AP}$ that contains the median from A.

h. Does the intersection of $\overleftrightarrow{CM}$ and $\overleftrightarrow{BN}$ lie on $\overleftrightarrow{AP}$?

i. Do the medians of this triangle intersect in one point?

8-4 THE SLOPES OF PERPENDICULAR LINES

Let l_1 be a line whose equation is $y = m_1x$ where m_1 is not equal to 0. Then $O(0, 0)$ and $A(1, m_1)$ are two points on the line.



$$\text{slope of } l_1 = \frac{m_1 - 0}{1 - 0} = \frac{m_1}{1} = m_1$$

Under a counterclockwise rotation of 90° about the origin, the image of $A(1, m_1)$ is $A'(-m_1, 1)$. Since $\angle AOA'$ is a right angle, $\overleftrightarrow{OA} \perp \overleftrightarrow{OA'}$.

Let l_2 be the line $\overleftrightarrow{OA'}$ through $A'(-m_1, 1)$ and $O(0, 0)$, and let the slope of l_2 be m_2. Then:

$$m_2 = \frac{0 - 1}{0 - (-m_1)} = \frac{-1}{0 + m_1} = -\frac{1}{m_1}$$

We have shown that when two lines through the origin are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

Is the rule that we found for the slopes of perpendicular lines through the origin true for perpendicular lines that do not intersect at the origin? We will show this by first establishing that translations preserve slope:

Theorem 8.2 Under a translation, slope is preserved, that is, if line l has slope m, then under a translation, the image of l also has slope m.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on line l. Then:

$$\text{slope of } l = \frac{y_2 - y_1}{x_2 - x_1}$$

Under a translation $T_{a,b}$, the images of P and Q have coordinates $P'(x_1 + a, y_1 + b)$ and $Q'(x_2 + a, y_2 + b)$. Therefore, the slope of l', the image of l, is

$$\begin{aligned}\text{slope of } l' &= \frac{(y_2 + b) - (y_1 + b)}{(x_2 + a) - (x_1 + a)} \\ &= \frac{y_2 - y_1 + b - b}{x_2 - x_1 + a - a} \\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$

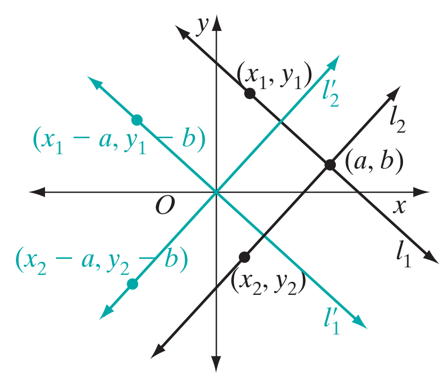
As a simple application of Theorem 8.2, we can show that the slopes of any two perpendicular lines are negative reciprocals of each other:

Theorem 8.3a

If two non-vertical lines are perpendicular, then the slope of one is the negative reciprocal of the other.

Proof: Let l_1 and l_2 be two perpendicular lines that intersect at (a, b). Under the translation $(x, y) \rightarrow (x - a, y - b)$, the image of (a, b) is $(0, 0)$.

Theorem 8.2 tells us that if the slope of l_1 is m, then the slope of its image, l_1', is m. Since l_1 and l_2 are perpendicular, their images, l_1' and l_2', are also perpendicular because translations preserve angle measure. Using what we established at the beginning of the section, since the slope of l_1' is m, the slope of l_2' is $-\frac{1}{m}$. Slope is preserved under a translation. Therefore, the slope of l_2 is $-\frac{1}{m}$.



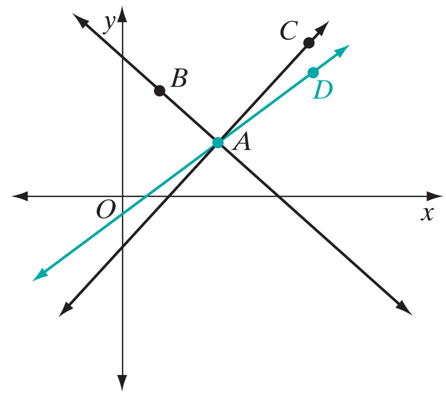
The proof of Theorem 8.3a is called a **transformational proof** since it uses transformations to prove the desired result.

Is the converse of Theorem 8.3a also true? To demonstrate that it is, we need to show that if the slope of one line is the negative reciprocal of the slope of the other, then the lines are perpendicular. We will use an indirect proof.

Theorem 8.3b If the slopes of two lines are negative reciprocals, then the two lines are perpendicular.

Given Two lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, that intersect at A. The slope of $\overleftrightarrow{AB}$ is m and the slope of $\overleftrightarrow{AC}$ is $-\frac{1}{m}$, the negative reciprocal of m.

Prove $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$



Proof

Statements	Reasons
1. $\overleftrightarrow{AC}$ is not perpendicular to $\overleftrightarrow{AB}$.	**1.** Assumption.
2. Construct $\overleftrightarrow{AD}$ perpendicular to $\overleftrightarrow{AB}$ at A.	**2.** At a point on a given line, one and only one perpendicular can be drawn.
3. Slope of $\overleftrightarrow{AB}$ is m.	**3.** Given.
4. The slope of $\overleftrightarrow{AD}$ is $-\frac{1}{m}$.	**4.** If two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.
5. The slope of $\overleftrightarrow{AC}$ is $-\frac{1}{m}$.	**5.** Given.
6. A, C, and D are on the same line, that is, $\overleftrightarrow{AC}$ and $\overleftrightarrow{AD}$ are the same line.	**6.** Three points lie on the same line if and only if the slope of the segment joining two of the points is equal to the slope of a segment joining another pair of these points.
7. $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$	**7.** Contradiction (steps 1, 6).

We can restate Theorems 8.3a and 8.3b as a biconditional.

Theorem 8.3 Two non-vertical lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other.

EXAMPLE 1

What is the slope of l_2, the line perpendicular to l_1, if the equation of l_1 is $x + 2y = 4$?

Solution (1) Solve the equation of l_1 for y:

$$x + 2y = 4$$
$$2y = -x + 4$$
$$y = -\tfrac{1}{2}x + 2$$

(2) Find the slope of l_1:

$$y = -\tfrac{1}{2}x + 2$$

slope

(3) Find the slope of l_2, the negative reciprocal of the slope of l_1:

slope of $l_2 = 2$

Answer The slope of l_2 is 2.

EXAMPLE 2

Show that $\triangle ABC$ is a right triangle if its vertices are $A(1, 1)$, $B(4, 3)$, and $C(2, 6)$.

Solution The slope of $\overline{AB}$ is $\frac{3-1}{4-1} = \frac{2}{3}$.

The slope of $\overline{BC}$ is $\frac{3-6}{4-2} = \frac{-3}{2} = -\frac{3}{2}$.

The slope of $\overline{AC}$ is $\frac{1-6}{1-2} = \frac{-5}{-1} = 5$.

The slope of $\overline{BC}$ is the negative reciprocal of the slope of $\overline{AB}$. Therefore, $\overleftrightarrow{BC}$ is perpendicular to $\overleftrightarrow{AB}$, $\angle B$ is a right angle, and $\triangle ABC$ is a right triangle.

Exercises

Writing About Mathematics

1. Explain why the slope of a line perpendicular to the line whose equation is $x = 5$ cannot be found by using the negative reciprocal of the slope of that line.
2. The slope of a line l is $-\frac{1}{3}$. Kim said that the slope of a line perpendicular to l is $-\frac{1}{-\frac{1}{3}}$. Santos said that the slope of a line perpendicular to l is 3. Who is correct? Explain your answer.

Developing Skills

In 3–12: **a.** Find the slope of the given line. **b.** Find the slope of the line perpendicular to the given line.

3. $y = 4x - 7$

4. $y = x + 2$

5. $x + y = 8$

6. $2x - y = 3$

7. $3x = 5 - 2y$

8. through (1, 1) and (5, 3)

9. through (0, 4) and (2, 0)

10. y-intercept -2 and x-intercept 4

11. through (4, 4) and (4, -2)

12. parallel to the x-axis through (5, 1)

In 13–16, find the equation of the line through the given point and perpendicular to the given line.

13. $\left(-\frac{1}{2}, -2\right)$; $2x + 7y = -15$

14. (0, 0); $-2x + 4y = 12$

15. (7, 3); $y = -\frac{1}{3}x - 3$

16. (2, -2); $y = 1$

17. Is the line whose equation is $y = -3x + 5$ perpendicular to the line whose equation is $3x + y = 6$?

18. Two perpendicular lines have the same y-intercept. If the equation of one of these lines is $y = \frac{1}{2}x - 1$, what is the equation of the other line?

19. Two perpendicular lines intersect at (2, -1). If $x - y = 3$ is the equation of one of these lines, what is the equation of the other line?

20. Write an equation of the line that intersects the y-axis at (0, -1) and is perpendicular to the line whose equation is $x + 2y = 6$.

In 21–24, the coordinates of the endpoints of a line segment are given. For each segment, find the equation of the line that is the perpendicular bisector of the segment.

21. $A(2, 2)$, $B(-1, 1)$

22. $A\left(-\frac{1}{2}, 3\right)$, $B\left(\frac{3}{2}, 1\right)$

23. $A(3, -9)$, $B(3, 9)$

24. $A(-4, -1)$, $B(3, -3)$

Applying Skills

25. The vertices of DEF triangle are $D(-3, 4)$, $E(-1, -2)$, and $F(3, 2)$.

a. Find an equation of the altitude from vertex D of $\triangle DEF$.

b. Is the altitude from D also the median from D? Explain your answer.

c. Prove that $\triangle DEF$ is isosceles.

26. If a four-sided polygon has four right angles, then it is a rectangle. Prove that if the vertices of a polygon are $A(3, -2)$, $B(5, 1)$, $C(-1, 5)$, and $D(-3, 2)$, then $ABCD$ is a rectangle.

27. The vertices of $\triangle ABC$ are $A(2, 2)$, $B(6, 6)$ and $C(6, 0)$.

a. What is the slope of $\overline{AB}$?

b. Write an equation for the perpendicular bisector of $\overline{AB}$.

c. What is the slope of $\overline{BC}$?

d. Write an equation for the perpendicular bisector of $\overline{BC}$.

e. What is the slope of $\overline{AC}$?

f. Write an equation for the perpendicular bisector of $\overline{AC}$.

g. Verify that the perpendicular bisectors of $\triangle ABC$ intersect in one point.

28. The coordinates of the vertices of $\triangle ABC$ are $A(-2, 0)$, $B(4, 0)$, and $C(0, 4)$.

a. Write the equation of each altitude of the triangle.

b. Find the coordinates of the point of intersection of these altitudes.

29. The coordinates of $\triangle LMN$ are $L(2, 5)$, $M(2, -3)$, and $N(7, -3)$.

a. Using the theorems of Section 7-6, prove that $\angle L$ and $\angle N$ are acute angles.

b. List the sides of the triangles in order, starting with the shortest.

c. List the angles of the triangles in order, starting with the smallest.

Hands-On Activity

The following activity may be completed using graph paper, pencil, compass, and straightedge, or geometry software.

In the exercises of Section 6-5, we saw how a translation in the horizontal direction can be achieved by a composition of two line reflections in vertical lines and a translation in the vertical direction can be achieved by a composition of two line reflections in horizontal lines. In this activity, we will see how any translation is a composition of two line reflections.

STEP 1. Draw any point A on a coordinate plane.

STEP 2. Translate the point A to its image A' under the given translation, $T_{a,b}$.

STEP 3. Draw line $\overleftrightarrow{AA'}$.

STEP 4. Draw any line l_1 perpendicular to $\overleftrightarrow{AA'}$.

STEP 5. Reflect the point A in line l_1. Let its image be A''.

STEP 6. Let l_2 be the line that is the perpendicular bisector of $\overline{A'A''}$.

Result: The given translation, $T_{a,b}$, is the composition of the two line reflections, r_{l_1} and r_{l_2}, in that order, that is, $T_{a,b} = r_{l_1} \circ r_{l_2}$. (Recall that r_{l_1} is performed first.)

For **a**–**c**, using the procedure above write the equations of two lines under which reflections in the two lines are equal to the given translation. Check your answers using the given coordinates.

a. $T_{4,4}$
$D(0, 0)$, $E(5, 3)$, $F(2, 2)$

b. $T_{-3,2}$
$D(-1, 2)$, $E(-5, 3)$, $F(5, 6)$

c. $T_{-1,-3}$
$D(6, 6)$, $E(-2, -5)$, $F(1, 6)$

8-5 COORDINATE PROOF

Many of the proofs that we did in the preceding chapters can also be done using coordinates. In particular, we can use the formula for slope, the slopes of perpendicular lines, and the coordinates of the midpoint of a line segment presented in this chapter to prove theorems about triangles. In later chapters we will use coordinates to prove theorems about polygons, parallel lines, and distances.

There are two types of proofs in coordinate geometry:

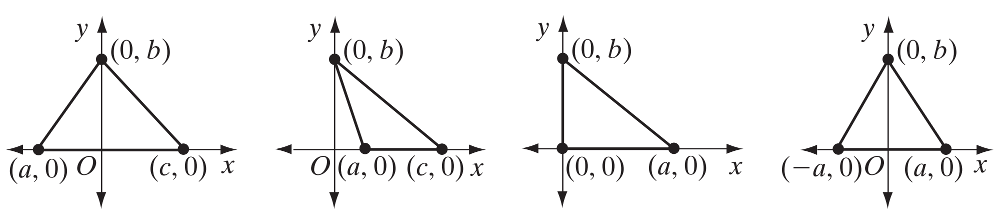
1. *Proofs Involving Special Cases.* When the coordinates of the endpoints of a segment or the vertices of a polygon are given as ordered pairs of numbers, we are proving something about a specific segment or polygon. (See Example 1.)

2. *Proofs of General Theorems.* When the given information is a figure that represents a particular type of polygon, we must state the coordinates of its vertices in general terms using variables. Those coordinates can be any convenient variables. Since it is possible to use a transformation that is an isometry to move a triangle without changing its size and shape, a geometric figure can be placed so that one of its sides is a segment of the x-axis. If two line segments or adjacent sides of a polygon are perpendicular, they can be represented as segments of the x-axis and the y-axis. (See Example 2.)

To prove that line segments bisect each other, show that the coordinates of the midpoints of each segment are the same ordered pair, that is, are the same point.

To prove that two lines are perpendicular to each other, show that the slope of one line is the negative reciprocal of the slope of the other.

The vertices shown in the diagrams below can be used when working with triangles.



The triangle with vertices $(a, 0)$, $(0, b)$, $(c, 0)$ can be any triangle. It is convenient to place one side of the triangle on the x-axis and the vertex opposite that side on the y-axis. The triangle can be acute if a and c have opposite signs or obtuse if a and c have the same sign.

A triangle with vertices at $(a, 0)$, $(0, 0)$, $(0, b)$ is a right triangle because it has a right angle at the origin.

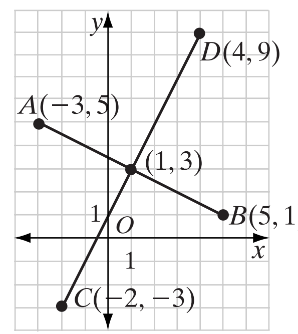
A triangle with vertices at $(-a, 0)$, $(0, b)$, $(a, 0)$ is isosceles because the altitude and the median are the same line segment.

When a general proof involves the midpoint of a segment, it is helpful to express the coordinates of the endpoints of the segment as variables divisible by 2. For example, if we had written the coordinates of the vertices of a right triangle as $(d, 0)$, $(0, e)$ and $(f, 0)$, we could simply let $d = 2a$, $e = 2b$, and $f = 2c$ so that the coordinates would be $(2a, 0)$, $(0, 2b)$ and $(2c, 0)$. The coordinates of midpoints of the sides of this triangle would be simpler using these coordinates.

EXAMPLE 1

Prove that $\overline{AB}$ and $\overline{CD}$ bisect each other and are perpendicular to each other if the coordinates of the endpoints of these segments are $A(-3, 5)$, $B(5, 1)$, $C(-2, -3)$, and $D(4, 9)$.

Solution This is a proof involving a special case.



The midpoint of $\overline{AB}$ is $\left(\frac{-3+5}{2}, \frac{5+1}{2}\right) = \left(\frac{2}{2}, \frac{6}{2}\right) = (1, 3)$.

The midpoint of $\overline{CD}$ is $\left(\frac{-2+4}{2}, \frac{-3+9}{2}\right) = \left(\frac{2}{2}, \frac{6}{2}\right) = (1, 3)$.

The slope of $\overline{AB}$ is $\frac{5-1}{-3-5} = \frac{4}{-8} = -\frac{1}{2}$.

The slope of $\overline{CD}$ is $\frac{9-(-3)}{4-(-2)} = \frac{12}{6} = 2$.

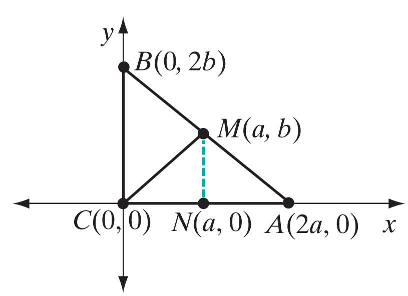
$\overline{AB}$ and $\overline{CD}$ bisect each other because they have a common midpoint, $(1, 3)$. $\overline{AB}$ and $\overline{CD}$ are perpendicular because the slope of one is the negative reciprocal of the slope of the other.

EXAMPLE 2

Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices.

Given: Right triangle ABC whose vertices are $A(2a, 0)$, $B(0, 2b)$, and $C(0, 0)$. Let M be the midpoint of the hypotenuse $\overline{AB}$.

Prove: $AM = BM = CM$



Proof This is a proof of a general theorem. Since it is a right triangle, we can place one vertex at the origin, one side of the triangle on the x-axis, and a second side on the y-axis so that these two sides form the right angle. We will use coordinates that are divisible by 2 to simplify computation of midpoints.

(1) The midpoint of a line segment is a point of that line segment that separates the line segment into two congruent segments. Therefore, $\overline{AM} \cong \overline{BM}$.

(2) The coordinates of M are $\left(\frac{2a + 0}{2}, \frac{0 + 2b}{2}\right) = (a, b)$.

(3) From M, draw a vertical segment that intersects $\overline{AC}$ at N. The x-coordinate of N is a because it is on the same vertical line as M. The y-coordinate of N is 0 because it is on the same horizontal line as A and C. The coordinates of N are $(a, 0)$.

(4) The midpoint of $\overline{AC}$ is $\left(\frac{0 + 2a}{2}, \frac{0 + 0}{2}\right) = (a, 0)$. N is the midpoint of $\overline{AC}$ and $\overline{AN} \cong \overline{NC}$.

(5) The vertical segment $\overline{MN}$ is perpendicular to the horizontal segment $\overline{AC}$. Perpendicular lines are two lines that intersect to form right angles. Therefore, $\angle ANM$ and $\angle CMN$ are right angles. All right angles are congruent, so $\angle ANM \cong \angle CMN$. Also, $\overline{MN} \cong \overline{MN}$.

(6) Then, $\triangle AMN \cong \triangle CMN$ by SAS (steps 4 and 5).

(7) $\overline{AM} \cong \overline{CM}$ because they are corresponding parts of congruent triangles.

(8) $\overline{AM} \cong \overline{BM}$ (step 1) and $\overline{AM} \cong \overline{CM}$ (step 7). Therefore, $\overline{AM} \cong \overline{BM} \cong \overline{CM}$ or $AM = BM = CM$. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices. ◻

Exercises

Writing About Mathematics

1. Ryan said that $(a, 0)$, $(0, b)$, $(c, 0)$ can be the vertices of any triangle but if $\frac{a}{b} = -\frac{b}{c}$, then the triangle is a right triangle. Do you agree with Ryan? Explain why or why not.

2. Ken said that $(a, 0)$, $(0, b)$, $(c, 0)$ can be the vertices of any triangle but if a and c have the same sign, then the triangle is obtuse. Do you agree with Ken? Explain why or why not.

Developing Skills

3. The coordinates of the endpoints of $\overline{AB}$ are $A(0, -2)$ and $B(4, 6)$. The coordinates of the endpoints of $\overline{CD}$ are $C(-4, 5)$ and $D(8, -1)$. Using the midpoint formula, show that the line segments bisect each other.

4. The vertices of polygon $ABCD$ are $A(2, 2)$, $B(5, -2)$, $C(9, 1)$, and $D(6, 5)$. Prove that the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and bisect each other using the midpoint formula.

5. The vertices of a triangle are $L(0, 1)$, $M(2, 5)$, and $N(6, 3)$.

   a. Find the coordinates K, the midpoint of the base, $\overline{LN}$.

   b. Show that $\overline{MK}$ is an altitude from M to $\overline{LN}$.

   c. Using parts **a** and **b**, prove that $\triangle LMN$ is isosceles.

6. The vertices of $\triangle ABC$ are $A(1, 7)$, $B\ (9, 3)$, and $C(3, 1)$.

   a. Prove that $\triangle ABC$ is a right triangle.

   b. Which angle is the right angle?

   c. Which side is the hypotenuse?

   d. What are the coordinates of the midpoint of the hypotenuse?

   e. What is the equation of the median from the vertex of the right angle to the hypotenuse?

   f. What is the equation of the altitude from the vertex of the right angle to the hypotenuse?

   g. Is the triangle an isosceles right triangle? Justify your answer using parts **e** and **f**.

7. The coordinates of the vertices of $\triangle ABC$ are $A(-4, 0)$, $B(0, 8)$, and $C(12, 0)$.

   a. Draw the triangle on graph paper.

   b. Find the coordinates of the midpoints of each side of the triangle.

   c. Find the slope of each side of the triangle.

   d. Write the equation of the perpendicular bisector of each side of the triangle.

   e. Find the coordinates of the circumcenter of the triangle.

8. A rhombus is a quadrilateral with four congruent sides. The vertices of rhombus $ABCD$ are $A(2, -3)$, $B(5, 1)$, $C(10, 1)$, and $D(7, -3)$.

   a. Prove that the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

   b. Prove that the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other.

Applying Skills

9. The vertices of rectangle $PQRS$ are $P(0, 0)$, $Q(a, 0)$, $R(a, b)$, and $S(0, b)$. Use congruent triangles to prove that the diagonals of a rectangle are congruent, that is, $\overline{PR} \cong \overline{QS}$.

10. The vertices of square $EFGH$ are $E(0, 0)$, $F(a, 0)$, $G(a, a)$, and $H(0, a)$. Prove that the diagonals of a square, $\overline{EG}$ and $\overline{FH}$, are the perpendicular bisectors of each other using the midpoint formula.

11. Use congruent triangles to prove that $(0, 0)$, $(2a, 0)$, and (a, b) are the vertices of an isosceles triangle. (*Suggestion:* Draw the altitude from (a, b).)

12. Use a translation to prove that $(-a, 0)$, $(0, b)$, and $(a, 0)$ are the vertices of an isosceles triangle. (*Hint:* A translation will let you use the results of Exercise 11.)

13. The coordinates of the vertices of $\triangle ABC$ are $A(0, 0)$, $B(2a, 2b)$, and $C(2c, 2d)$.

a. Find the coordinates of E, the midpoint of $\overline{AB}$ and of F, the midpoint of $\overline{AC}$.

b. Prove that the slope of $\overline{EF}$ is equal to the slope of $\overline{BC}$.

14. The endpoints of segment $\overline{AB}$ are $(-a, 0)$ and $(a, 0)$.

a. Use congruent triangles to show that $P(0, b)$ and $Q(0, c)$ are both equidistant from the endpoints of $\overline{AB}$.

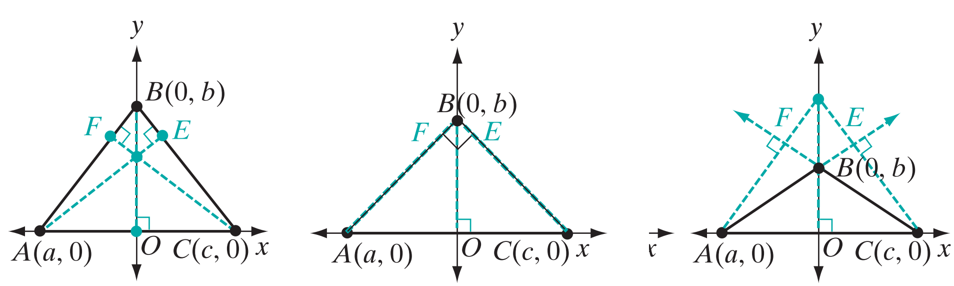
b. Show that $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

8-6 CONCURRENCE OF THE ALTITUDES OF A TRIANGLE

The postulates of the coordinate plane and the statements that we have proved about the slopes of perpendicular lines make it possible for us to prove that the three altitudes of a triangle are *concurrent*, that is, they intersect in one point.

Theorem 8.4 The altitudes of a triangle are concurrent.

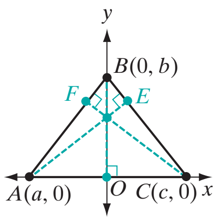
Proof: We can place the triangle anywhere in the coordinate plane. We will place it so that $\overline{AC}$ lies on the x-axis and B lies on the y-axis. Let $A(a, 0)$, $B(0, b)$, and $C(c, 0)$ be the vertices of $\triangle ABC$. Let $\overline{AE}$ be the altitude from A to $\overline{BC}$, $\overline{BO}$ be the altitude from B to $\overline{AC}$, and $\overline{CF}$ be the altitude from C to $\overline{AB}$.

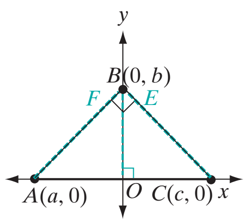


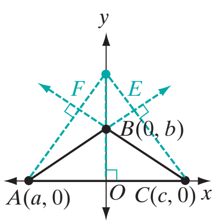
$\angle B$ is acute $\angle B$ is right $\angle B$ is obtuse

In the figures, $\angle A$ and $\angle C$ are acute angles.

We will show that altitudes $\overline{AE}$ and $\overline{BO}$ intersect in the same point as altitudes $\overline{CF}$ and $\overline{BO}$.







Intersection of altitudes $\overline{AE}$ and $\overline{BO}$

1. The slope of side $\overline{BC}$ is

$$\frac{0 - b}{c - 0} = -\frac{b}{c}.$$

2. The slope of altitude $\overline{AE}$, which is perpendicular to $\overline{BC}$, is $\frac{c}{b}$.

3. The equation of $\overline{AE}$ is

$$\frac{y - 0}{x - a} = \frac{c}{b} \text{ or } y = \frac{c}{b}x - \frac{ac}{b}.$$

4. Since $\overline{AC}$ is a horizontal line, $\overline{BO}$ is a vertical line, a segment of the y-axis since B is on the y-axis.

5. The equation of $\overleftrightarrow{BO}$ is $x = 0$.

6. The coordinates of the intersection of $\overline{AE}$ and $\overline{BO}$ can be found by finding the common solution of their equations: $y = \frac{c}{b}x - \frac{ac}{b}$ and $x = 0$.

7. Since one of the equations is $x = 0$, replace x by 0 in the other equation:

$$\begin{aligned} y &= \frac{c}{b}x - \frac{ac}{b} \\ &= \frac{c}{b}(0) - \frac{ac}{b} \\ &= -\frac{ac}{b} \end{aligned}$$

8. The coordinates of the intersection of $\overline{AE}$ and $\overline{BO}$ are $\left(0, -\frac{ac}{b}\right)$.

Intersection of altitudes $\overline{CF}$ and $\overline{BO}$

1. The slope of side $\overline{AB}$ is

$$\frac{0 - b}{a - 0} = -\frac{b}{a}.$$

2. The slope of altitude $\overline{CF}$, which is perpendicular to $\overline{AB}$, is $\frac{a}{b}$.

3. The equation of $\overline{CF}$ is

$$\frac{y - 0}{x - c} = \frac{a}{b} \text{ or } y = \frac{a}{b}x - \frac{ac}{b}.$$

4. Since $\overline{AC}$ is a horizontal line, $\overline{BO}$ is a vertical line, a segment of the y-axis since B is on the y-axis.

5. The equation of $\overleftrightarrow{BO}$ is $x = 0$.

6. The coordinates of the intersection of $\overline{CF}$ and $\overline{BO}$ can be found by finding the common solution of their equations: $y = \frac{a}{b}x - \frac{ac}{b}$ and $x = 0$.

7. Since one of the equations is $x = 0$, replace x by 0 in the other equation:

$$\begin{aligned} y &= \frac{a}{b}x - \frac{ac}{b} \\ &= \frac{a}{b}(0) - \frac{ac}{b} \\ &= -\frac{ac}{b} \end{aligned}$$

8. The coordinates of the intersection of $\overline{CF}$ and $\overline{BO}$ are $\left(0, -\frac{ac}{b}\right)$.

The altitudes of a triangle are concurrent at $\left(0, -\frac{ac}{b}\right)$. ❑

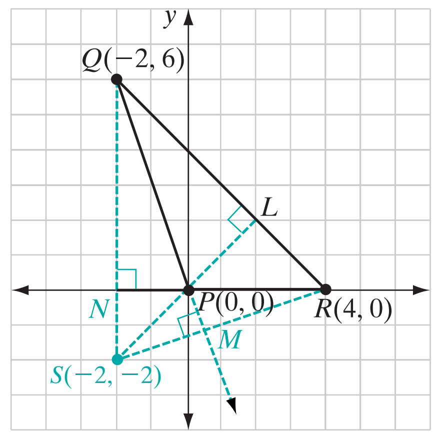
Note: If B is a right angle, $\overline{CB}$ is the altitude from C to $\overline{AB}$, $\overline{AB}$ is the altitude from A to $\overline{CB}$, and $\overline{BO}$ is the altitude from B to $\overline{AC}$. The intersection of these three altitudes is $B(0, b)$.

The point where the altitudes of a triangle intersect is called the **orthocenter**.

EXAMPLE 1

The coordinates of the vertices of $\triangle PQR$ are $P(0, 0)$, $Q(-2, 6)$, and $R(4, 0)$. Find the coordinates of the orthocenter of the triangle.

Solution Let $\overline{PL}$ be the altitude from P to $\overline{QR}$.



The slope of $\overline{QR}$ is $\frac{6 - 0}{-2 - 4} = \frac{6}{-6} = -1$.

The slope of $\overline{PL}$ is 1.

The equation of $\overline{PL}$ is $\frac{y - 0}{x - 0} = 1$ or $y = x$.

Let $\overline{QN}$ be the altitude from Q to $\overline{PR}$. The point of intersection, N, is on the line determined by P and R.

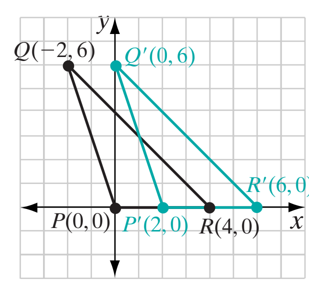
The slope of $\overline{PR}$ is 0 since $\overline{PR}$ is a horizontal line. Therefore, $\overline{QN}$ is a segment of a vertical line that has no slope.

The equation of $\overline{QN}$ is $x = -2$.

The intersection S of the altitudes $\overline{QN}$ and $\overline{PL}$ is the common solution of the equations $x = -2$ and $y = x$. Therefore, the coordinates of the intersection S are $(-2, -2)$. By Theorem 8.4, point S is the orthocenter of the triangle or the point where the altitudes are concurrent.

Answer The orthocenter of $\triangle PQR$ is $S(-2, -2)$.

Alternative Solution Use the result of the proof given in this section. The coordinates of the point of intersection of the altitudes are $\left(0, -\frac{ac}{b}\right)$. In order for this result to apply, Q must lie on the y-axis and P and R must lie on the x-axis. Since P and R already lie on the x-axis, we need only to use a translation to move Q in the horizontal direction to the y-axis. Use the translation $(x, y) \rightarrow (x + 2, y)$:



$$P(0, 0) \rightarrow P'(2, 0) \qquad Q(-2, 6) \rightarrow Q'(0, 6) \qquad R(4, 0) \rightarrow R'(6, 0)$$

Therefore,

$$A(a, 0) = P'(2, 0) \text{ or } a = 2$$
$$B(0, b) = Q'(0, 6) \text{ or } b = 6$$
$$C(c, 0) = R'(6, 0) \text{ or } c = 6.$$

The coordinates of S', the point at which the altitudes of $\triangle P'R'Q'$ intersect, are

$$\left(0, -\frac{ac}{b}\right) = \left(0, -\frac{2(6)}{6}\right) = (0, -2)$$

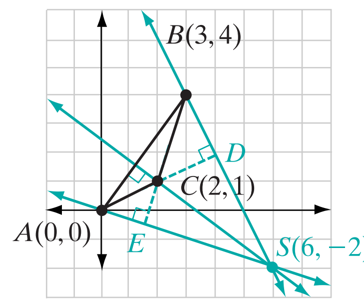
The intersection of the altitudes of $\triangle PQR$ is S, the preimage of $S'(0, -2)$ under the translation $(x, y) \rightarrow (x + 2, y)$. Therefore, the coordinates of S are $(-2, -2)$.

Answer The orthocenter of $\triangle PQR$ is $(-2, -2)$.

EXAMPLE 2

The coordinates of the vertices of $\triangle ABC$ are $A(0, 0)$, $B(3, 4)$ and $C(2, 1)$. Find the coordinates of the orthocenter of the triangle.

Solution The slope of $\overline{AC}$ is $\frac{1-0}{2-0} = \frac{1}{2}$.



Let $\overline{BD}$ be the altitude from B to $\overline{AC}$.

The slope of altitude $\overline{BD}$ is -2.

The equation of line $\overleftrightarrow{BD}$ is

$$\frac{4-y}{3-x} = -2$$
$$4 - y = -2(3 - x)$$
$$-y = -6 + 2x - 4$$
$$y = -2x + 10$$

The slope of $\overline{BC}$ is $\frac{4-1}{3-2} = \frac{3}{1} = 3$.

Let $\overline{AE}$ be the altitude from A to $\overline{BC}$.

The slope of altitude $\overline{AE}$ is $-\frac{1}{3}$.

The equation of line $\overleftrightarrow{AE}$ is

$$\frac{y-0}{x-0} = -\frac{1}{3}$$
$$y = -\frac{1}{3}x$$

The orthocenter S is the common solution of the equations $y = -2x + 10$ and $y = -\frac{1}{3}x$.

$$-2x + 10 = -\frac{1}{3}x$$
$$-1\tfrac{2}{3}x = -10$$
$$-\frac{5}{3}x = -10$$
$$x = -10\left(\frac{-3}{5}\right)$$
$$x = 6$$

The x-coordinate is 6; the y-coordinate is $-\frac{1}{3}(6) = -2$.

Answer The coordinates of the orthocenter are $(6, -2)$

Exercises

Writing About Mathematics

In 1–2, the vertices of $\triangle ABC$ are $A(a, 0)$, $B(0, b)$, and $C(c, 0)$, as shown in the diagrams of the proof of Theorem 8.4. Assume that $b > 0$.

1. Esther said that if A is to the left of the origin and C is to the right of the origin, then the point of intersection of the altitudes is above the origin. Do you agree with Esther? Explain why or why not.

2. Simon said that if $\angle A$ is an obtuse angle, and both A and C are to the right of the origin, then the point of intersection of the altitudes is above the origin. Do you agree with Simon? Explain why or why not.

Developing Skills

3. The coordinates of $\triangle DEF$ are $D(-9, 0)$, $E(0, 12)$, and $F(16, 0)$.

a. Show that $\triangle DEF$ is a right triangle.

b. Show that E is the orthocenter of the triangle.

In 4–9, find the coordinates of the orthocenter of each triangle with the given vertices.

4. $A(-2, 0)$, $B(0, 6)$, $C(3, 0)$

5. $D(-12, 0)$, $E(0, 8)$, $F(6, 0)$

6. $L(7, 2)$, $M(2, 12)$, $N(11, 2)$

7. $P(-3, 4)$, $Q(1, 8)$, $R(3, 4)$

8. $G(-5, 2)$, $H(4, 8)$, $I(5, 1)$

9. $J(0, -3)$, $K(3, 4)$, $L(2, -1)$

Applying Skills

10. Two of the vertices of $\triangle ABC$ are $A(-3, 0)$ and $C(6, 0)$. The altitudes from these vertices intersect at $P(0, 3)$.

a. $\overleftrightarrow{AB}$ is a line through A, perpendicular to $\overline{CP}$. Write the equation of $\overleftrightarrow{AB}$.

b. $\overleftrightarrow{CB}$ is a line through C, perpendicular to $\overline{AP}$. Write the equation of $\overleftrightarrow{CB}$.

c. Find B, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$.

d. Write an equation of the line that contains the altitude from B to $\overline{AC}$.

e. Show that P is a point on that line.

11. Two of the vertices of $\triangle ABC$ are $A(2, -2)$ and $C(5, 5)$. The altitudes from these vertices intersect at $P(1, 1)$.

a. $\overleftrightarrow{AB}$ is a line through A, perpendicular to $\overline{CP}$. Write the equation of $\overleftrightarrow{AB}$.

b. $\overleftrightarrow{CB}$ is a line through C, perpendicular to $\overline{AP}$. Write the equation of $\overleftrightarrow{CB}$.

c. Find B, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$.

d. Write an equation of the line that contains the altitude from B to $\overline{AC}$.

e. Show that P is a point on that line.

Hands-On Activity

In this activity we will use a compass and a straightedge to construct the orthocenters of various triangles.

For each triangle: **a.** Graph the triangle on paper or using geometry software. **b.** Using a compass, straightedge, and pencil, or geometry software, construct the orthocenter. (If using the computer, you are only allowed to use the point, line segment, line, and circle creation tools of your geometry software and no other software tools.)

(1) $A(-3, 0), B(0, 2), C(4, 0)$

(2) $D(-4, -7), E(0, 5), F(3, -1)$

(3) $G(-4, 2), H(6, 0), I(0, -4)$

CHAPTER SUMMARY

Definitions to Know

- The **slope**, m, of a line that passes through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 \neq x_2$, is the ratio of the difference of the y-values of these points to the difference of the corresponding x-values.

$$\text{slope of } \overleftrightarrow{AB} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Three or more lines are **concurrent** if they intersect in one point.
- The **orthocenter** of a triangle is the point at which the altitudes of a triangle intersect.

Postulate

8.1 A, B, and C lie on the same line if and only if the slope of $\overline{AB}$ is equal to the slope of $\overline{BC}$.

Theorems

8.1 If the endpoints of a line segment are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint of the segment are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

8.2 Under a translation, slope is preserved, that is, if line l has slope m, then under a translation, the image of l also has slope m.

8.3 Two non-vertical lines are perpendicular if and only if the slope of one is the negative reciprocal of the other.

8.4 The altitudes of a triangle are concurrent.

VOCABULARY

8-1 Slope • Δx • Δy

8-2 y-intercept • x-intercept • Point-slope form of an equation • Slope-intercept form of an equation

8-3 Midpoint formula

8-4 Transformational proof

8-6 Orthocenter

REVIEW EXERCISES

In 1–3, the coordinates of the endpoints of a line segment are given. **a.** Find the coordinates of each midpoint. **b.** Find the slope of each segment. **c**. Write an equation of the line that is determined by each pair of points.

1. $(0, 0)$ and $(6, -4)$ **2.** $(-3, 2)$ and $(7, -4)$ **3.** $(1, -1)$ and $(5, -5)$

4. The coordinates of P are $(-2, 5)$ and the coordinates of Q are $(6, 1)$.

a. What is the slope of $\overleftrightarrow{PQ}$?

b. What is the equation of $\overleftrightarrow{PQ}$?

c. What are the coordinates of the midpoint of $\overline{PQ}$?

d. What is the equation of the perpendicular bisector of $\overline{PQ}$?

5. The vertices of $\triangle RST$ are $R(-2, -2)$, $S(1, 4)$, and $T(7, 1)$.

a. Show that $\triangle RST$ is a right triangle.

b. Find the coordinates of the midpoint of $\overline{RT}$.

c. Write the equation of the line that contains the median from S.

d. Show that the median of the triangle from S is also the altitude from S.

e. Prove that $\triangle RST$ is an isosceles triangle.

6. Two of the vertices of $\triangle ABC$ are $A(1, 2)$ and $B(9, 6)$. The slope of $\overline{AC}$ is 1 and the slope of $\overline{BC}$ is $-\frac{1}{3}$. What are the coordinates of B?

7. The vertices of $\triangle DEF$ are $D(1, 1)$, $E(5, 5)$, $F(-1, 5)$.

a. Find the coordinates of the midpoint of each side of the triangle.

b. Find the slope of each side of the triangle.

c. Find the slope of each altitude of the triangle.

d. Write an equation of the perpendicular bisector of each side of the triangle.

e. Show that the three perpendicular bisectors intersect in a point and find the coordinates of that point.

8. The vertices of $\triangle ABC$ are $A(-7, 1)$, $B(5, -3)$, and $C(-3, 5)$.

a. Prove that $\triangle ABC$ is a right triangle.

b. Let M be the midpoint of $\overline{AB}$ and N be the midpoint of $\overline{AC}$. Prove that $\triangle AMN \cong \triangle CMN$ and use this result to show that M is equidistant from the vertices of $\triangle ABC$.

9. The vertices of $\triangle DEF$ are $D(-2, -3)$, $E(5, 0)$, and $F(-2, 3)$

a. Find the coordinates of M, the midpoint of $\overline{DF}$.

b. Show that $\overline{DE} \cong \overline{FE}$.

10. The coordinates of the vertices of quadrilateral $ABCD$ are $A(-5, -4)$, $B(1, -6)$, $C(-1, -2)$, and $D(-4, -1)$. Show that $ABCD$ has a right angle.

Exploration

The following exploration may be completed using graph paper or using geometry software.

We know that the area of a triangle is equal to one-half the product of the lengths of the base and height. We can find the area of a right triangle in the coordinate plane if the base and height are segments of the x-axis and y-axis. The steps that follow will enable us to find the area of any triangle in the coordinate plane. For example, find the area of a triangle if the vertices have the coordinates $(2, 5)$, $(5, 9)$, and $(8, -2)$.

STEP 1. Plot the points and draw the triangle.

STEP 2. Through the vertex with the smallest x-coordinate, draw a vertical line.

STEP 3. Through the vertex with the largest x-coordinate, draw a vertical line.

STEP 4. Through the vertex with the smallest y-coordinate, draw a horizontal line.

STEP 5. Through the vertex with the largest y-coordinate, draw a horizontal line.

STEP 6. The triangle is now enclosed by a rectangle with horizontal and vertical sides. Find the length and width of the rectangle and the area of the rectangle.

STEP 7. The rectangle is separated into four triangles: the given triangle and three triangles that have a base and altitude that are a horizontal and a vertical line. Find the area of these three triangles.

STEP 8. Find the sum of the three areas found in step 7. Subtract this sum from the area of the rectangle. The difference is the area of the given triangle.

Repeat steps 1 through 8 for each of the triangles with the given vertices.

a. $(-2, 0)$, $(3, 7)$, $(6, -1)$ **b.** $(-3, -6)$, $(0, 4)$, $(5, -1)$ **c.** $(0, -5)$, $(6, 6)$, $(2, 5)$

Can you use this procedure to find the area of the quadrilateral with vertices at $(0, 2)$, $(5, -2)$, $(5, 3)$, and $(2, 5)$?

CUMULATIVE REVIEW Chapters 1–8

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. When the coordinates of A are $(2, -3)$ and of B are $(2, 7)$, AB equals
(1) 4 (2) -4 (3) 10 (4) -10

2. The slope of the line whose equation is $2x - y = 4$ is
(1) $\frac{1}{2}$ (2) 2 (3) -2 (4) 4

3. Which of the following is an example of the associative property for multiplication?
(1) $3(2 + 5) = 3(5 + 2)$
(2) $3(2 \cdot 5) = (3 \cdot 2) \cdot 5$
(3) $3(2 + 5) = 3(2) + 5$
(4) $3 + (2 + 5) = (3 + 2) + 5$

4. The endpoints of $\overline{AB}$ are $A(0, 6)$ and $B(-4, 0)$. The coordinates of the midpoint of $\overline{AB}$ are
(1) $(-2, 3)$ (2) $(2, -3)$ (3) $(-2, -3)$ (4) $(2, 3)$

5. In isosceles triangle DEF, $DE = DF$. Which of the following is true?
(1) $\angle D \cong \angle E$
(2) $\angle F \cong \angle E$
(3) $\angle D \cong \angle F$
(4) $\angle D \cong \angle E \cong \angle F$

6. The slope of line l is $\frac{2}{3}$. The slope of a line perpendicular to l is
(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) $\frac{3}{2}$ (4) $-\frac{3}{2}$

7. The coordinates of two points are $(0, 6)$ and $(3, 0)$. The equation of the line through these points is
(1) $y = 2x + 6$
(2) $y = -2x + 6$
(3) $y = \frac{1}{2}x + 3$
(4) $y = -\frac{1}{2}x + 3$

8. The converse of the statement "If two angles are right angles then they are congruent" is
(1) If two angles are congruent then they are right angles.
(2) If two angles are not right angles then they are not congruent.
(3) Two angles are congruent if and only if they are right angles.
(4) Two angles are congruent if they are right angles.

9. The measure of an angle is twice the measure of its supplement. The measure of the smaller angle is
(1) 30 (2) 60 (3) 90 (4) 120

10. Under a reflection in the y-axis, the image of $(4, -2)$ is
(1) $(-4, 2)$ (2) $(4, -2)$ (3) $(-4, -2)$ (4) $(2, 4)$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. a. Draw $\triangle ABC$ in the coordinate plane if the coordinates of A are $(-3, 1)$, of B are $(1, -5)$, and of C are $(5, 2)$.

b. Under a translation, the image of A is $A'(1, -1)$. Draw $\triangle A'B'C'$, the image of $\triangle ABC$ under this translation, and give the coordinates of B' and C'.

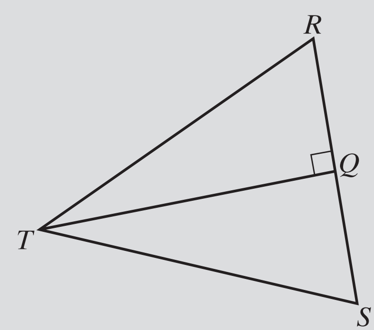
c. If this translation can be written as $(x, y) \to (x + a, y + b)$, what are the values of a and b?

12. The coordinates of the vertices of $\triangle DEF$ are $D(-1, 6)$, $E(3, 3)$, and $F(1, 2)$. Is $\triangle DEF$ a right triangle? Justify your answer.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. In the diagram, $\overrightarrow{TQ}$ bisects $\angle RTS$ and $\overline{TQ} \perp \overline{RS}$. Prove that $\triangle RST$ is isosceles.



14. The following statements are true:

- If Evanston is not the capital of Illinois, then Chicago is not the capital.
- Springfield is the capital of Illinois or Chicago is the capital of Illinois.
- Evanston is not the capital of Illinois.

Use the laws of logic to prove that Springfield is the capital of Illinois.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Find the coordinates of the point of intersection of the lines whose equations are $y = 3x - 1$ and $x + 2y = 5$.

16. *Given:* $\overline{CGF}$ and $\overline{DGE}$ bisect each other at G.

Prove: $\overline{CD} \cong \overline{FE}$