

CHAPTER

7

CHAPTER TABLE OF CONTENTS

- 7-1 Basic Inequality Postulates
- 7-2 Inequality Postulates Involving Addition and Subtraction
- 7-3 Inequality Postulates Involving Multiplication and Division
- 7-4 An Inequality Involving the Lengths of the Sides of a Triangle
- 7-5 An Inequality Involving an Exterior Angle of a Triangle
- 7-6 Inequalities Involving Sides and Angles of a Triangle
- Chapter Summary
- Vocabulary
- Review Exercises
- Cumulative Review

GEOMETRIC INEQUALITIES

Euclid's Proposition 20 of Book I of the *Elements* states, "In any triangle, two sides taken together in any manner are greater than the remaining one."

The Epicureans, a group of early Greek philosophers, ridiculed this theorem, stating that it is evident even to a donkey since if food is placed at one vertex of a triangle and the donkey at another, the donkey will make his way along one side of the triangle rather than traverse the other two, to get to the food. But no matter how evident the truth of a statement may be, it is important that it be logically established in order that it may be used in the proof of theorems that follow. Many of the inequality theorems of this chapter depend on this statement for their proof.

7-1 BASIC INEQUALITY POSTULATES

Each time the athletes of the world assemble for the Olympic Games, they attempt to not only perform better than their competitors at the games but also to surpass previous records in their sport. News commentators are constantly comparing the winning time of a bobsled run or a 500-meter skate with the world records and with individual competitors' records.

In previous chapters, we have studied pairs of congruent lines and pairs of congruent angles that have equal measures. But pairs of lines and pairs of angles are often not congruent and have unequal measures. In this chapter, we will apply the basic inequality principles that we used in algebra to the lengths of line segments and the measures of angles. These inequalities will enable us to formulate many important geometric relationships.

Postulate Relating a Whole Quantity and Its Parts

In Chapter 3 we stated postulates of equality. Many of these postulates suggest related postulates of inequality.

Consider the partition postulate:

► **A whole is equal to the sum of all its parts.**

This corresponds to the following postulate of inequality:

Postulate 7.1

A whole is greater than any of its parts.

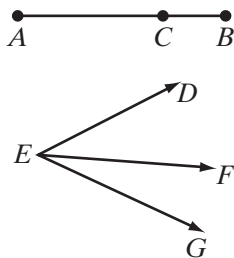
In arithmetic: Since $14 = 9 + 5$, then $14 > 9$ and $14 > 5$.

In algebra: If a , b , and c represent positive numbers and $a = b + c$, then $a > b$ and $a > c$.

In geometry: The lengths of line segments and the measures of angles are positive numbers.

Consider these two applications:

- If \overline{ACB} is a line segment, then $AB = AC + CB$, $AB > AC$, and $AB > CB$.
- If $\angle DEF$ and $\angle FEG$ are adjacent angles, $m\angle DEG = m\angle DEF + m\angle FEG$, $m\angle DEG > m\angle DEF$, and $m\angle DEG > m\angle FEG$.



Transitive Property

Consider this statement of the transitive property of equality:

► **If a , b , and c are real numbers such that $a = b$ and $b = c$, then $a = c$.**

This corresponds to the following **transitive property of inequality**:

Postulate 7.2

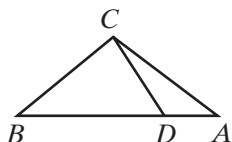
If a , b , and c are real numbers such that $a > b$ and $b > c$, then $a > c$.

In arithmetic: If $12 > 7$ and $7 > 3$, then $12 > 3$.

In algebra: If $5x + 1 > 2x$ and $2x > 16$, then $5x + 1 > 16$.

In geometry: If $BA > BD$ and $BD > BC$, then $BA > BC$.

Also, if $m\angle BCA > m\angle BCD$ and $m\angle BCD > m\angle BAC$, then $m\angle BCA > m\angle BAC$.



Substitution Postulate

Consider the substitution postulate as it relates to equality:

► **A quantity may be substituted for its equal in any statement of equality.**

Substitution also holds for inequality, as demonstrated in the following postulate:

Postulate 7.3

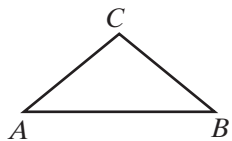
A quantity may be substituted for its equal in any statement of inequality.

In arithmetic: If $10 > 2 + 5$ and $2 + 5 = 7$, then $10 > 7$.

In algebra: If $5x + 1 > 2y$ and $y = 4$, then $5x + 1 > 2(4)$.

In geometry: If $AB > BC$ and $BC = AC$, then $AB > AC$.

Also, if $m\angle C > m\angle A$ and $m\angle A = m\angle B$, then $m\angle C > m\angle B$.



The Trichotomy Postulate

We know that if x represents the coordinate of a point on the number line, then x can be a point to the left of 3 when $x < 3$, x can be the point whose coordinate is 3 if $x = 3$, or x can be a point to the right of 3 if $x > 3$. We can state this as a postulate that we call the **trichotomy postulate**, meaning that it is divided into three cases.

Postulate 7.4

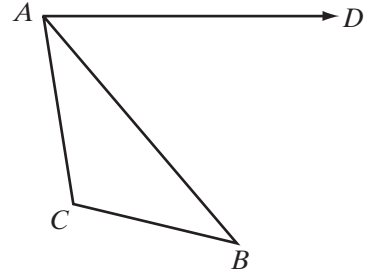
Given any two quantities, a and b , one and only one of the following is true:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$

EXAMPLE 1

Given: $m\angle DAC = m\angle DAB + m\angle BAC$ and
 $m\angle DAB > m\angle ABC$

Prove: $m\angle DAC > m\angle ABC$



Proof	Statements	Reasons
	1. $m\angle DAC = m\angle DAB + m\angle BAC$	1. Given.
	2. $m\angle DAC > m\angle DAB$	2. A whole is greater than any of its parts.
	3. $m\angle DAB > m\angle ABC$	3. Given.
	4. $m\angle DAC > m\angle ABC$	4. Transitive property of inequality.

EXAMPLE 2

Given: Q is the midpoint of \overline{PS} and
 $RS < QS$.



Prove: $RS < PQ$

Proof	Statements	Reasons
	1. Q is the midpoint of \overline{PS} .	1. Given.
	2. $\overline{PQ} \cong \overline{QS}$	2. The midpoint of a line segment is the point that divides the segment into two congruent segments.
	3. $PQ = QS$.	3. Congruent segments have equal measures.
	4. $RS < QS$	4. Given.
	5. $RS < PQ$	5. Substitution postulate.

Exercises

Writing About Mathematics

1. Is inequality an equivalence relation? Explain why or why not.
2. Monica said that when $AB > BC$ is false, $AB < BC$ must be true. Do you agree with Monica? Explain your answer.

Developing Skills

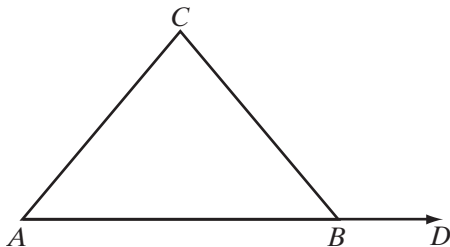
In 3–12: **a.** Draw a diagram to illustrate the hypothesis and tell whether each conclusion is true or false. **b.** State a postulate or a definition that justifies your answer.

3. If \overline{ADB} is a line segment, then $DB < AB$.
4. If D is not on \overline{AC} , then $CD + DA < CA$.
5. If $\angle BCD + \angle DCA = \angle BCA$, then $m\angle BCD < m\angle BCA$.
6. If \overrightarrow{DB} and \overrightarrow{DA} are opposite rays with point C not on \overrightarrow{DB} or \overrightarrow{DA} , then $m\angle BDC + m\angle CDA = 180$.
7. If \overrightarrow{DB} and \overrightarrow{DA} are opposite rays and $m\angle BDC > 90$, then $m\angle CDA > 90$.
8. If \overline{ADB} is a line segment, then $DA > BD$, or $DA = BD$, or $DA < BD$.
9. If $AT > AS$ and $AS > AR$, then $AT > AR$.
10. If $m\angle 1 > m\angle 2$ and $m\angle 2 > m\angle 3$, then $m\angle 1 > m\angle 3$.
11. If $SR > KR$ and $SR = TR$, then $TR > KR$.
12. If $m\angle 3 < m\angle 2$ and $m\angle 2 = m\angle 1$, then $m\angle 3 < m\angle 1$.

Applying Skills

13. Given: $\triangle ABC$ is isosceles, $\overline{AC} \cong \overline{BC}$,
 $m\angle CBD > m\angle CBA$

Prove: $m\angle CBD > m\angle A$



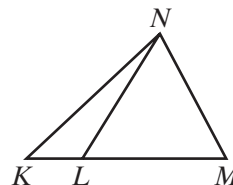
14. Given: \overline{PQRS} and $PQ = RS$

Prove: **a.** $PR > PQ$ **b.** $PR > RS$



In 15 and 16, use the figure to the right.

15. If \overline{KLM} and $\overline{LM} \cong \overline{NM}$, prove that $KM > NM$.
16. If $KM > KN$, $KN > NM$, and $NM = NL$, prove that $KM > NL$.



7-2 INEQUALITY POSTULATES INVOLVING ADDITION AND SUBTRACTION

Postulates of equality and examples of inequalities involving the numbers of arithmetic can help us to understand the inequality postulates presented here.

Consider the addition postulate:

- **If equal quantities are added to equal quantities, then the sums are equal.**

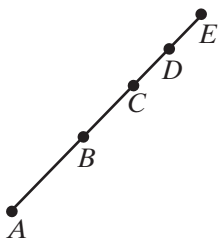
Addition of inequalities requires two cases:

Postulate 7.5

If equal quantities are added to unequal quantities, then the sums are unequal in the same order.

Postulate 7.6

If unequal quantities are added to unequal quantities in the same order, then the sums are unequal in the same order.



- In arithmetic: Since $12 > 5$, then $12 + 3 > 5 + 3$ or $15 > 8$.
 Since $12 > 5$ and $3 > 2$, then $12 + 3 > 5 + 2$ or $15 > 7$.
- In algebra: If $x - 5 > 10$, then $x - 5 + 5 > 10 + 5$ or $x > 15$.
 If $x - 5 > 10$ and $5 > 3$, then $x - 5 + 5 > 10 + 3$ or $x > 13$.
- In geometry: If \overline{ABCD} and $AB > CD$, then $AB + BC > BC + CD$ or $AC > BD$.
 If \overline{ABCDE} , $AB > CD$, and $BC > DE$, then $AB + BC > CD + DE$ or $AC > CE$.

We can subtract equal quantities from unequal quantities without changing the order of the inequality, but the result is uncertain when we subtract unequal quantities from unequal quantities.

Consider the subtraction postulate:

- **If equal quantities are subtracted from equal quantities, then the differences are equal.**

Subtraction of inequalities is restricted to a single case:

Postulate 7.7

If equal quantities are subtracted from unequal quantities, then the differences are unequal in the same order.

However, when unequal quantities are subtracted from unequal quantities, the results may or may not be unequal and the order of the inequality may or may not be the same.

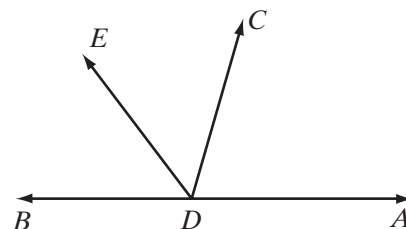
For example:

- $5 > 2$ and $4 > 1$, but it is not true that $5 - 4 > 2 - 1$ since $1 = 1$.
- $12 > 10$ and $7 > 1$, but it is not true that $12 - 7 > 10 - 1$ since $5 < 9$.
- $12 > 10$ and $2 > 1$, and it is true that $12 - 2 > 10 - 1$ since $10 > 9$.

EXAMPLE I

Given: $m\angle BDE < m\angle CDA$

Prove: $m\angle BDC < m\angle EDA$

**Proof**

Statements	Reasons
1. $m\angle BDE < m\angle CDA$	1. Given.
2. $m\angle BDE + m\angle EDC < m\angle EDC + m\angle CDA$	2. If equal quantities are added to unequal quantities, then the sums are unequal in the same order.
3. $m\angle BDC = m\angle BDE + m\angle EDC$	3. The whole is equal to the sum of its parts.
4. $m\angle EDA = m\angle EDC + m\angle CDA$	4. The whole is equal to the sum of its parts.
5. $m\angle BDC < m\angle EDA$	5. Substitution postulate for inequalities.



Exercises

Writing About Mathematics

- Dana said that $13 > 11$ and $8 > 3$. Therefore, $13 - 8 < 11 - 3$ tells us that if unequal quantities are subtracted from unequal quantities, the difference is unequal in the opposite order. Do you agree with Dana? Explain why or why not.
- Ella said that if unequal quantities are subtracted from equal quantities, then the differences are unequal in the opposite order. Do you agree with Ella? Explain why or why not.

Developing Skills

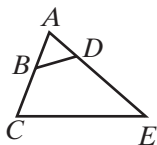
In 3–10, in each case use an inequality postulate to prove the conclusion.

- If $10 > 7$, then $18 > 15$.
- If $x + 3 > 12$, then $x > 9$.
- If $8 > 6$ and $5 > 3$, then $13 > 9$.
- If $y > 8$, then $y - 1 > 7$.
- If $4 < 14$, then $15 < 25$.
- If $y - 5 < 5$, then $y < 10$.
- If $7 < 12$, then $5 < 10$.
- If $a = b$, then $180 - a > 90 - b$.

Applying Skills

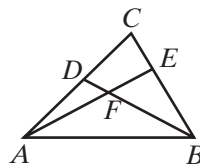
11. Given: $AB = AD$, $BC < DE$

Prove: $AC < AE$



12. Given: $AE > BD$, $AF = BF$

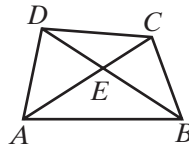
Prove: $FE > FD$



13. Given: $m\angle DAC > m\angle DBC$ and $AE = BE$

Prove: a. $m\angle EAB = m\angle EBA$

b. $m\angle DAB > m\angle CBA$



- In August, Blake weighed more than Caleb. In the next two months, Blake and Caleb had each gained the same number of pounds. Does Blake still weigh more than Caleb? Justify your answer.
- In December, Blake weighed more than Andre. In the next two months, Blake lost more than Andre lost. Does Blake still weigh more than Andre? Justify your answer.

7-3 INEQUALITY POSTULATES INVOLVING MULTIPLICATION AND DIVISION

Since there are equality postulates for multiplication and division similar to those of addition and subtraction, we would expect that there are inequality postulates for multiplication and division similar to those of addition and subtraction. Consider these examples that use both positive and negative numbers.

If $9 > 3$, then $9(4) > 3(4)$ or $36 > 12$.	If $-9 < -3$, then $-9(4) < -3(4)$ or $-36 < -12$.
If $1 < 5$, then $1(3) < 5(3)$ or $3 < 15$.	If $-1 > -5$, then $-1(3) > -5(3)$ or $-1 > -15$.
If $9 > 3$, then $9(-4) < 3(-4)$ or $-36 < -12$.	If $-9 < -3$, then $-9(-4) > -3(-4)$ or $36 > 12$.
If $1 < 5$, then $1(-3) > 5(-3)$ or $-3 > -15$.	If $-1 > -5$, then $-1(-3) < -5(-3)$ or $3 < 15$.

Notice that in the top four examples, we are multiplying by positive numbers and the order of the inequality does not change. In the bottom four examples, we are multiplying by negative numbers and the order of the inequality does change.

These examples suggest the following postulates of inequality:

Postulate 7.8

If unequal quantities are multiplied by positive equal quantities, then the products are unequal in the same order.

Postulate 7.9

If unequal quantities are multiplied by negative equal quantities, then the products are unequal in the opposite order.

Since we know that division by $a \neq 0$ is the same as multiplication by $\frac{1}{a}$ and that a and $\frac{1}{a}$ are always either both positive or both negative, we can write similar postulates for division of inequalities.

Postulate 7.10

If unequal quantities are divided by positive equal quantities, then the quotients are unequal in the same order.

Postulate 7.11

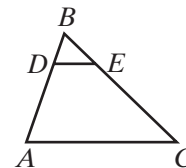
If unequal quantities are divided by negative equal quantities, then the quotients are unequal in the opposite order.

Care must be taken when using inequality postulates involving multiplication and division because multiplying or dividing by a *negative* number will reverse the order of the inequality.

EXAMPLE 1

Given: $BA = 3BD$, $BC = 3BE$, and $BE > BD$

Prove: $BC > BA$

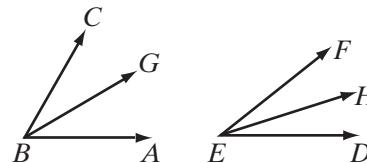


Proof	Statements	Reasons
	1. $BE > BD$	1. Given.
	2. $3BE > 3BD$	2. If unequal quantities are multiplied by positive equal quantities, then the products are unequal in the same order.
	3. $BC = 3BE$, $BA = 3BD$	3. Given.
	4. $BC > BA$	4. Substitution postulate for inequalities. ■

EXAMPLE 2

Given: $m\angle ABC > m\angle DEF$, \overrightarrow{BG} bisects $\angle ABC$, \overrightarrow{EH} bisects $\angle DEF$.

Prove: $m\angle ABG > m\angle DEH$



Proof An angle bisector separates the angle into two congruent parts. Therefore, the measure of each part is one-half the measure of the angle that was bisected, so $m\angle ABG = \frac{1}{2}m\angle ABC$ and $m\angle DEH = \frac{1}{2}m\angle DEF$.

Since we are given that $m\angle ABC > m\angle DEF$, $\frac{1}{2}m\angle ABC > \frac{1}{2}m\angle DEF$ because if unequal quantities are multiplied by positive equal quantities, the products are unequal in the same order. Therefore, by the substitution postulate for inequality, $m\angle ABG > m\angle DEH$. ■

Exercises

Writing About Mathematics

- Since $1 < 2$, is it always true that $a < 2a$? Explain why or why not.
- Is it always true that if $a > b$ and $c > d$, then $ac > bd$? Justify your answer.

Developing Skills

In 3–8, in each case state an inequality postulate to prove the conclusion.

- If $8 > 7$, then $24 > 21$.
- If $8 > 6$, then $4 > 3$.
- If $\frac{x}{2} > -4$, then $-x < 8$.
- If $30 < 35$, then $-6 > -7$.
- If $3x > 15$, then $x > 5$.
- If $\frac{y}{6} < 3$, then $y < 18$.

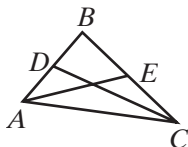
In 9–17: If a , b , and c are positive real numbers such that $a > b$ and $b > c$, tell whether each relationship is *always true*, *sometimes true*, or *never true*. If the statement is always true, state the postulate illustrated. If the statement is sometimes true, give one example for which it is true and one for which it is false. If the statement is never true, give one example for which it is false.

- $ac > bc$
- $a - c > b - c$
- $\frac{a}{c} > \frac{b}{c}$
- $a + c > b + c$
- $a - b > b - c$
- $-ac > -bc$
- $c - a > c - b$
- $\frac{c}{a} > \frac{c}{b}$
- $a > c$

Applying Skills

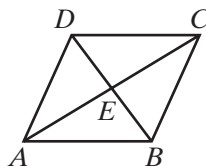
18. Given: $BD < BE$, D is the midpoint of \overline{BA} , E is the midpoint of \overline{BC} .

Prove: $BA < BC$



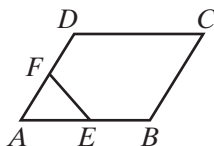
19. Given: $m\angle DBA > m\angle CAB$,
 $m\angle CBA = 2m\angle DBA$,
 $m\angle DAB = 2m\angle CAB$

Prove: $m\angle CBA > m\angle DAB$



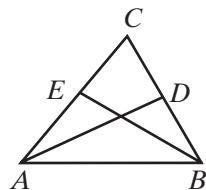
20. Given: $AB > AD$, $AE = \frac{1}{2}AB$,
 $AF = \frac{1}{2}AD$

Prove: $AE > AF$



21. Given: $m\angle CAB < m\angle CBA$, \overline{AD} bisects $\angle CAB$, \overline{BE} bisects $\angle CBA$.

Prove: $m\angle DAB < m\angle EBA$

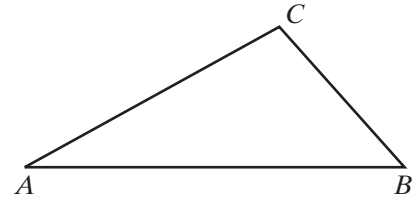


7-4 AN INEQUALITY INVOLVING THE LENGTHS OF THE SIDES OF A TRIANGLE

The two quantities to be compared are often the lengths of line segments or the distances between two points. The following postulate was stated in Chapter 4.

► **The shortest distance between two points is the length of the line segment joining these two points.**

The vertices of a triangle are three noncollinear points. The length of \overline{AB} is AB , the shortest distance from A to B . Therefore, $AB < AC + CB$. Similarly, $BC < BA + AC$ and $AC < AB + BC$. We have just proved the following theorem, called the **triangle inequality theorem**:



Theorem 7.1

The length of one side of a triangle is less than the sum of the lengths of the other two sides.

In the triangle shown above, $AB > AC > BC$. To show that the lengths of three line segments can be the measures of the sides of a triangle, we must show that the length of any side is less than the sum of the other two lengths of the other two sides.

EXAMPLE 1

Which of the following may be the lengths of the sides of a triangle?

- (1) 4, 6, 10 (2) 8, 8, 16 (3) 6, 8, 16 (4) 10, 12, 14

Solution The length of a side of a triangle must be less than the sum of the lengths of the other two sides. If the lengths of the sides are $a < b < c$, then $a < b$ means that $a < b + c$ and $b < c$ means that $b < c + a$. Therefore, we need only test the longest side.

- (1) Is $10 < 4 + 6$? **No**
 (2) Is $16 < 8 + 8$? **No**
 (3) Is $16 < 6 + 8$? **No**
 (4) Is $14 < 10 + 12$? **Yes Answer**

EXAMPLE 2

Two sides of a triangle have lengths 3 and 7. Find the range of possible lengths of the third side.

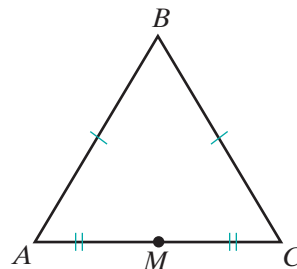
- Solution**
- (1) Let s = length of third side of triangle.
 - (2) Of the lengths 3, 7, and s , the longest side is either 7 or s .
 - (3) If the length of the longest side is s , then $s < 3 + 7$ or $s < 10$.
 - (4) If the length of the longest side is 7, then $7 < s + 3$ or $4 < s$.

Answer $4 < s < 10$ ■

EXAMPLE 3

Given: Isosceles triangle ABC with $AB = CB$ and M the midpoint of \overline{AC} .

Prove: $AM < AB$

**Proof**

Statements	Reasons
1. $AC < AB + CB$	1. The length of one side of a triangle is less than the sum of the lengths of the other two sides.
2. $AB = CB$	2. Given.
3. $AC < AB + AB$ or $AC < 2AB$	3. Substitution postulate for inequalities.
4. M is the midpoint of \overline{AC} .	4. Given.
5. $AM = MC$	5. Definition of a midpoint.
6. $AC = AM + MC$	6. Partition postulate.
7. $AC = AM + AM = 2AM$	7. Substitution postulate.
8. $2AM < 2AB$	8. Substitution postulate for inequality.
9. $AM < AB$	9. Division postulate for inequality. ■

Exercises**Writing About Mathematics**

- If 7, 12, and s are the lengths of three sides of a triangle, and s is not the longest side, what are the possible values of s ?
- If $a < b < c$ are any real numbers, is $a < b + c$ always true? Justify your answer.
 - If $a < b < c$ are the lengths of the sides of a triangle, is $a < b + c$ always true? Justify your answer.

Developing Skills

In 3–10, tell in each case whether the given lengths can be the measures of the sides of a triangle.

- | | | | |
|-------------------|--------------------|--------------------|---------------------|
| 3. 3, 4, 5 | 4. 5, 8, 13 | 5. 6, 7, 10 | 6. 3, 9, 15 |
| 7. 2, 2, 3 | 8. 1, 1, 2 | 9. 3, 4, 4 | 10. 5, 8, 11 |

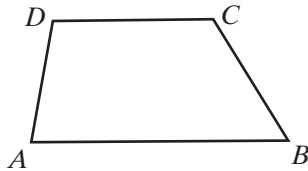
In 11–14, find values for r and t such that the inequality $r < s < t$ best describes s , the length of the third sides of a triangle for which the lengths of the other two sides are given.

- 2 and 4
- 12 and 31
- $\frac{13}{2}$ and $\frac{13}{2}$
- 9.6 and 12.5
- Explain why x , $2x$, and $3x$ cannot represent the lengths of the sides of a triangle.
- For what values of a can a , $a + 2$, $a - 2$ represent the lengths of the sides of a triangle? Justify your answer.

Applying Skills

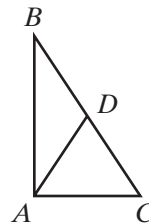
- 17.** Given: $ABCD$ is a quadrilateral.

Prove: $AD < AB + BC + CD$



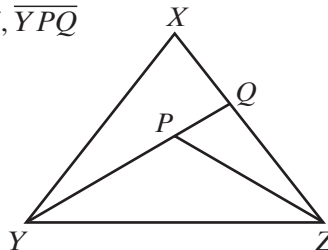
- 18.** Given: $\triangle ABC$ with D a point on \overline{BC} and $AD = DC$.

Prove: $AB < BC$



- 19.** Given: Point P in the interior of $\triangle XYZ$, \overline{YPQ}

Prove: $PY + PZ < XY + XZ$



Hands-On Activity

One side of a triangle has a length of 6. The lengths of the other two sides are integers that are less than or equal to 6.

- a. Cut one straw 6 inches long and two sets of straws to integral lengths of 1 inch to 6 inches. Determine which lengths can represent the sides of a triangle.

Or

Use geometry software to determine which lengths can represent the sides of a triangle.

- b. List all sets of three integers that can be the lengths of the sides of the triangle. For example,

$$\{6, 3, 5\}$$

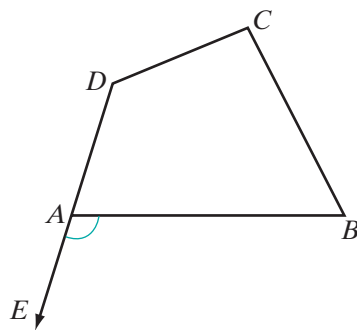
is one set of lengths.

- c. List all sets of three integers less than or equal to 6 that *cannot* be the lengths of the sides of the triangle.
- d. What patterns emerge in the results of parts **b** and **c**?

7-5 AN INEQUALITY INVOLVING AN EXTERIOR ANGLE OF A TRIANGLE

Exterior Angles of a Polygon

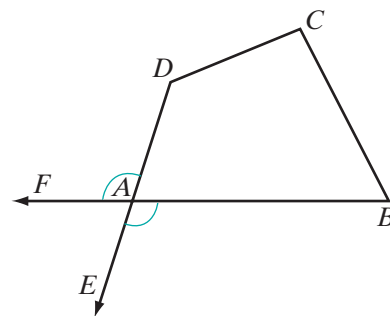
At each vertex of a polygon, an angle is formed that is the union of two sides of the polygon. Thus, for polygon $ABCD$, $\angle DAB$ is an angle of the polygon, often called an interior angle. If, at vertex A , we draw \overrightarrow{AE} , the opposite ray of \overrightarrow{AD} , we form $\angle BAE$, an exterior angle of the polygon at vertex A .



DEFINITION

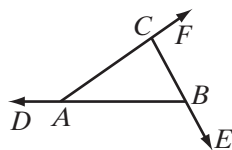
An **exterior angle of a polygon** is an angle that forms a linear pair with one of the interior angles of the polygon.

At vertex A , we can also draw \overrightarrow{AF} , the opposite ray of \overrightarrow{AB} , to form $\angle DAF$, another exterior angle of the polygon at vertex A . At each vertex of a polygon, two exterior angles can be drawn. Each of these exterior angles forms a linear pair with the interior angle at A , and the angles in a linear pair are supplementary. The two exterior angles at A are congruent angles because they are vertical angles. Either can be drawn as the exterior angle at A .



Exterior Angles of a Triangle

An exterior angle of a triangle is formed outside the triangle by extending a side of the triangle.



The figure to the left shows $\triangle ABC$ whose three interior angles are $\angle CAB$, $\angle ABC$, and $\angle BCA$. By extending each side of $\triangle ABC$, three exterior angles are formed, namely, $\angle DAC$, $\angle EBA$, and $\angle FCB$.

For each exterior angle, there is an **adjacent interior angle** and two **remote** or **nonadjacent interior angles**. For $\triangle ABC$, these angles are as follows:

Vertex	Exterior Angle	Adjacent Interior Angle	Nonadjacent Interior Angles
A	$\angle DAC$	$\angle CAB$	$\angle ABC$ and $\angle BCA$
B	$\angle EBA$	$\angle ABC$	$\angle CAB$ and $\angle BCA$
C	$\angle FCB$	$\angle BCA$	$\angle CAB$ and $\angle ABC$

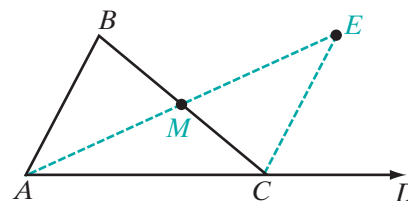
With these facts in mind, we are now ready to prove another theorem about inequalities in geometry called the **exterior angle inequality theorem**.

Theorem 7.2

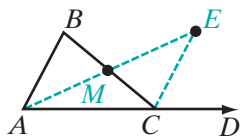
The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.

Given $\triangle ABC$ with exterior $\angle BCD$ at vertex C ;
 $\angle A$ and $\angle B$ are nonadjacent interior angles with respect to $\angle BCD$.

Prove $m\angle BCD > m\angle B$



Proof



Statements	Reasons
1. Let M be the midpoint of \overline{BC} .	1. Every line segment has one and only one midpoint.
2. Draw \overrightarrow{AM} , extending the ray through M to point E so that $\overline{AM} \cong \overline{EM}$.	2. Two points determine a line. A line segment can be extended to any length.
3. Draw \overline{EC} .	3. Two points determine a line.
4. $m\angle BCD = m\angle BCE + m\angle ECD$	4. A whole is equal to the sum of its parts.
5. $\overline{BM} \cong \overline{CM}$	5. Definition of midpoint.
6. $\overline{AM} \cong \overline{EM}$	6. Construction (step 2).
7. $\angle AMB \cong \angle EMC$	7. Vertical angles are congruent.
8. $\triangle AMB \cong \triangle EMC$	8. SAS (steps 5, 7, 6).
9. $\angle B \cong \angle MCE$	9. Corresponding parts of congruent triangles are congruent.
10. $m\angle BCD > m\angle MCE$	10. A whole is greater than any of its parts.
11. $m\angle BCD > m\angle B$	11. Substitution postulate for inequalities. ■

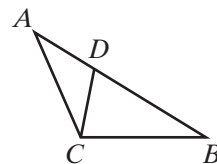
These steps prove that the measure of an exterior angle is greater than the measure of one of the nonadjacent interior angles, $\angle B$. A similar proof can be used to prove that the measure of an exterior angle is greater than the measure of the other nonadjacent interior angle, $\angle A$. This second proof uses N , the midpoint of \overline{AC} , a line segment \overline{BNG} with $\overline{BN} \cong \overline{NG}$, and a point F extending ray \overrightarrow{BC} through C .

The details of this proof will be left to the student. (See exercise 14.)

EXAMPLE I

The point D is on \overline{AB} of $\triangle ABC$.

- Name the exterior angle at D of $\triangle ADC$.
- Name two nonadjacent interior angles of the exterior angle at D of $\triangle ADC$.
- Why is $m\angle CDB > m\angle DCA$?
- Why is $AB > AD$?

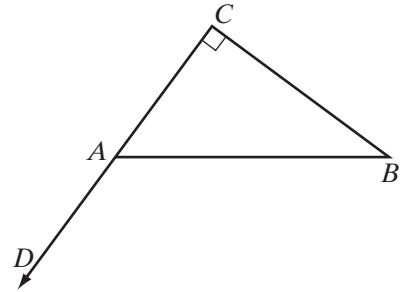


- Solution**
- $\angle CDB$
 - $\angle DCA$ and $\angle A$
 - The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.
 - The whole is greater than any of its parts. ■

EXAMPLE 2

Given: Right triangle ABC , $m\angle C = 90$,
 $\angle BAD$ is an exterior angle at A .

Prove: $\angle BAD$ is obtuse.



Proof	Statements	Reasons
	1. $\angle BAD$ is an exterior angle.	1. Given.
	2. $m\angle BAD > m\angle C$	2. Exterior angle inequality theorem.
	3. $m\angle C = 90$	3. Given.
	4. $m\angle BAD > 90$	4. Substitution postulate for inequalities.
	5. $m\angle BAD + m\angle BAC = 180$	5. If two angles form a linear pair, then they are supplementary.
	6. $180 > m\angle BAD$	6. The whole is greater than any of its parts.
	7. $180 > m\angle BAD > 90$	7. Steps 4 and 6.
	8. $\angle BAD$ is obtuse.	8. An obtuse angle is an angle whose degree measure is greater than 90 and less than 180. ■

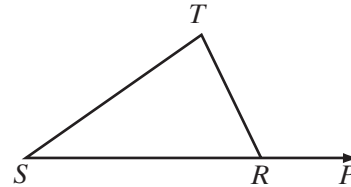
Exercises

Writing About Mathematics

1. Evan said that every right triangle has at least one exterior angle that is obtuse. Do you agree with Evan? Justify your answer.
2. Connor said that every right triangle has at least one exterior angle that is a right angle. Do you agree with Connor? Justify your answer.

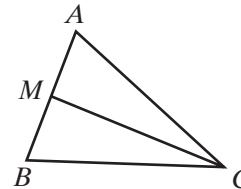
Developing Skills

3. a. Name the exterior angle at R .
- b. Name two nonadjacent interior angles of the exterior angle at R .



In 4–13, $\triangle ABC$ is scalene and \overline{CM} is a median to side \overline{AB} .

- a. Tell whether each given statement is *True* or *False*.
- b. If the statement is true, state the definition, postulate, or theorem that justifies your answer.

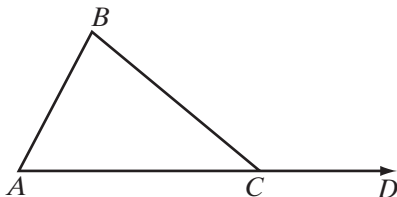


4. $AM = MB$
5. $m\angle ACB > m\angle ACM$
6. $m\angle AMC > m\angle ABC$
7. $AB > AM$
8. $m\angle CMB > m\angle ACM$
9. $m\angle CMB > m\angle CAB$
10. $BA > MB$
11. $m\angle ACM = m\angle BCM$
12. $m\angle BCA > m\angle MCA$
13. $m\angle BMC = m\angle AMC$

Applying Skills

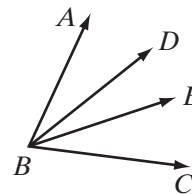
14. *Given:* $\triangle ABC$ with exterior $\angle BCD$ at vertex C ; $\angle A$ and $\angle B$ are nonadjacent interior angles with respect to $\angle BCD$.

Prove: $m\angle BCD > m\angle A$
(Complete the proof of Theorem 7.2).



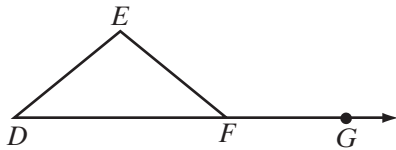
15. *Given:* $\angle ABD + \angle DBE = \angle ABE$
and $\angle ABE + \angle EBC = \angle ABC$

Prove: $m\angle ABD < m\angle ABC$



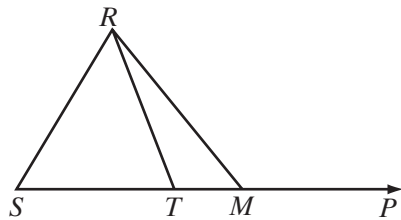
16. *Given:* Isosceles $\triangle DEF$ with $DE = FE$ and exterior $\angle EFG$

Prove: $m\angle EFG > m\angle EFD$



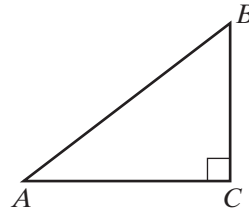
18. *Given:* $\triangle SMR$ with \overline{STM} extended through M to P

Prove: $m\angle RMP > m\angle SRT$



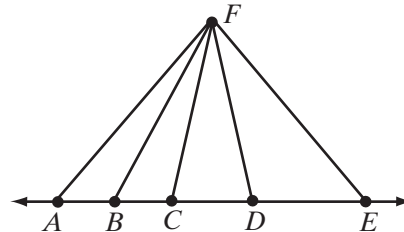
17. *Given:* Right $\triangle ABC$ with $m\angle C = 90$

Prove: $\angle A$ is acute.



19. *Given:* Point F not on $\overleftrightarrow{ABCDE}$ and $FC = FD$

Prove: $m\angle ABF > m\angle EDF$



7-6 INEQUALITIES INVOLVING SIDES AND ANGLES OF A TRIANGLE

We know that if the lengths of two sides of a triangle are equal, then the measures of the angles opposite these sides are equal. Now we want to compare the measures of two angles opposite sides of unequal length.

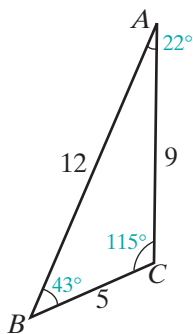
Let the measures of the sides of $\triangle ABC$ be $AB = 12$, $BC = 5$, and $CA = 9$.

Write the lengths in order: $12 > 9 > 5$

Name the sides in order: $AB > CA > BC$

Name the angles opposite these sides in order: $m\angle C > m\angle B > m\angle A$

Notice how the vertex of the angle opposite a side of the triangle is always the point that is not an endpoint of that side.



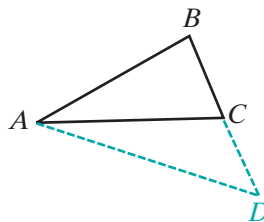
Theorem 7.3

If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal and the larger angle lies opposite the longer side.

To prove this theorem, we will extend the shorter side of a triangle to a length equal to that of the longer side, forming an isosceles triangle. We can then use the isosceles triangle theorem and the exterior angle inequality theorem to compare angle measures.

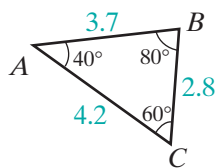
Given $\triangle ABC$ with $AB > BC$

Prove $m\angle ACB > m\angle BAC$



Proof

Statements	Reasons
1. $\triangle ABC$ with $AB > BC$.	1. Given.
2. Extend \overline{BC} through C to point D so that $BD = BA$.	2. A line segment may be extended to any length.
3. Draw \overline{AD} .	3. Two points determine a line.
4. $\triangle ABD$ is isosceles.	4. Definition of isosceles triangle.
5. $m\angle BAD = m\angle BDA$	5. Base angles of an isosceles triangle are equal in measure.
6. For $\triangle ACD$, $m\angle BCA > m\angle BDA$.	6. Exterior angle inequality theorem.
7. $m\angle BCA > m\angle BAD$	7. Substitution postulate for inequalities.
8. $m\angle BAD > m\angle BAC$	8. A whole is greater than any of its parts.
9. $m\angle BCA > m\angle BAC$	9. Transitive property of inequality.



The converse of this theorem is also true, as can be seen in this example:
Let the measures of the angles of $\triangle ABC$ be $m\angle A = 40$, $m\angle B = 80$, and $m\angle C = 60$.

Write the angle measures in order: $80 > 60 > 40$

Name the angles in order: $m\angle B > m\angle C > m\angle A$

Name the sides opposite these angles in order: $AC > AB > BC$

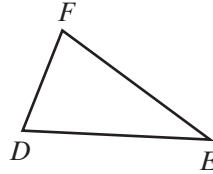
Theorem 7.4

If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal and the longer side lies opposite the larger angle.

We will write an indirect proof of this theorem. Recall that in an indirect proof, we assume the opposite of what is to be proved and show that the assumption leads to a contradiction.

Given $\triangle DEF$ with $m\angle D > m\angle E$

Prove $FE > FD$



Proof By the trichotomy postulate: $FE > FD$ or $FE = FD$ or $FE < FD$. We assume the negation of the conclusion, that is, we assume $FE \leq FD$. Therefore, either $FE = FD$ or $FE < FD$.

If $FE = FD$, then $m\angle D = m\angle E$ because base angles of an isosceles triangle are equal in measure. This contradicts the given premise, $m\angle D > m\angle E$. Thus, $FE = FD$ is a false assumption.

If $FE < FD$, then, by Theorem 7.3, we must conclude that $m\angle D < m\angle E$. This also contradicts the given premise that $m\angle D > m\angle E$. Thus, $FE < FD$ is also a false assumption.

Since $FE = FD$ and $FE < FD$ are both false, $FE > FD$ must be true and the theorem has been proved. ■

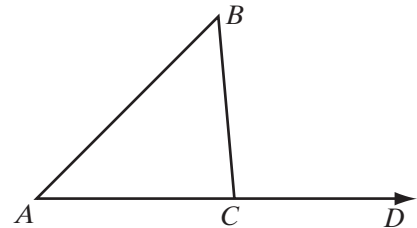
EXAMPLE 1

One side of $\triangle ABC$ is extended to D . If $m\angle A = 45$, $m\angle B = 50$, and $m\angle BCD = 95$, which is the longest side of $\triangle ABC$?

Solution The exterior angle and the interior angle at vertex C form a linear pair and are supplementary. Therefore:

$$\begin{aligned} m\angle BCA &= 180 - m\angle BCD \\ &= 180 - 95 \\ &= 85 \end{aligned}$$

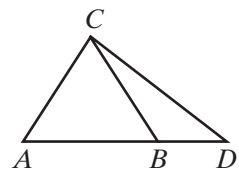
Since $85 > 50 > 45$, the longest side of the triangle is \overline{BA} , the side opposite $\angle BCA$. **Answer** ■

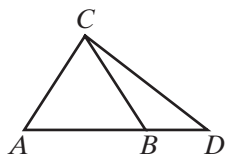


EXAMPLE 2

In $\triangle ADC$, \overline{CB} is drawn to \overline{AD} and $\overline{CA} \cong \overline{CB}$. Prove that $CD > CA$.

Proof Consider $\triangle CBD$. The measure of an exterior angle is greater than the measure of a nonadjacent interior angle, so $m\angle CBA > m\angle CDA$. Since $\overline{CA} \cong \overline{CB}$, $\triangle ABC$ is isosceles. The base angles of an isosceles triangle have equal measures, so $m\angle A = m\angle CBA$. A quantity may be substituted for its equal in an inequality, so $m\angle A > m\angle CDA$.





If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal and the longer side is opposite the larger angle. Therefore, $CD > AC$. ■

Exercises

Writing About Mathematics

1. **a.** Write the contrapositive of the statement “If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal.”
b. Is this contrapositive statement true?
2. The Isosceles Triangle Theorem states that if two sides of a triangle are congruent, then the angles opposite these sides are congruent.
a. Write the converse of the Isosceles Triangle Theorem.
b. How is this converse statement related to the contrapositive statement written in exercise 1?

Developing Skills

3. If $AB = 10$, $BC = 9$, and $CA = 11$, name the largest angle of $\triangle ABC$.
4. If $m\angle D = 60$, $m\angle E = 70$, and $m\angle F = 50$, name the longest side of $\triangle DEF$.

In 5 and 6, name the shortest side of $\triangle ABC$, using the given information.

5. In $\triangle ABC$, $m\angle C = 90$, $m\angle B = 35$, and $m\angle A = 55$.
6. In $\triangle ABC$, $m\angle A = 74$, $m\angle B = 58$, and $m\angle C = 48$.

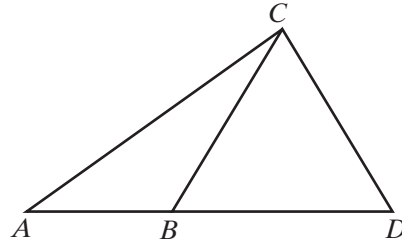
In 7 and 8, name the smallest angle of $\triangle ABC$, using the given information.

7. In $\triangle ABC$, $AB = 7$, $BC = 9$, and $AC = 5$.
8. In $\triangle ABC$, $AB = 5$, $BC = 12$, and $AC = 13$.
9. In $\triangle RST$, an exterior angle at R measures 80 degrees. If $m\angle S > m\angle T$, name the shortest side of the triangle.
10. If $\angle ABD$ is an exterior angle of $\triangle BCD$, $m\angle ABD = 118$, $m\angle D = 60$, and $m\angle C = 58$, list the sides of $\triangle BCD$ in order starting with the longest.
11. If $\angle EFH$ is an exterior angle of $\triangle FGH$, $m\angle EFH = 125$, $m\angle G = 65$, $m\angle H = 60$, list the sides of $\triangle FGH$ in order starting with the shortest.
12. In $\triangle RST$, $\angle S$ is obtuse and $m\angle R < m\angle T$. List the lengths of the sides of the triangle in order starting with the largest.

Applying Skills

13. Given: C is a point that is not on \overline{ABD} ,
 $m\angle ABC > m\angle CBD$.

Prove: $AC > BC$



14. Let $\triangle ABC$ be any right triangle with the right angle at C and hypotenuse \overline{AB} .
- Prove that $\angle A$ and $\angle B$ are acute angles.
 - Prove that the hypotenuse is the longest side of the right triangle.
15. Prove that every obtuse triangle has two acute angles.

CHAPTER SUMMARY

Definitions
to Know

- An **exterior angle of a polygon** is an angle that forms a linear pair with one of the interior angles of the polygon.
- Each exterior angle of a triangle has an **adjacent interior angle** and two **remote** or **nonadjacent interior angles**.

Postulates

- 7.1 A whole is greater than any of its parts.
 7.2 If a , b , and c are real numbers such that $a > b$ and $b > c$, then $a > c$.
 7.3 A quantity may be substituted for its equal in any statement of inequality.
 7.4 Given any two quantities, a and b , one and only one of the following is true: $a < b$, or $a = b$, or $a > b$.
 7.5 If equal quantities are added to unequal quantities, then the sums are unequal in the same order.
 7.6 If unequal quantities are added to unequal quantities in the same order, then the sums are unequal in the same order.
 7.7 If equal quantities are subtracted from unequal quantities, then the differences are unequal in the same order.
 7.8 If unequal quantities are multiplied by positive equal quantities, then the products are unequal in the same order.
 7.9 If unequal quantities are multiplied by negative equal quantities, then the products are unequal in the opposite order.
 7.10 If unequal quantities are divided by positive equal quantities, then the quotients are unequal in the same order.
 7.11 If unequal quantities are divided by negative equal quantities, then the quotients are unequal in the opposite order.

Theorems

- 7.1 The length of one side of a triangle is less than the sum of the lengths of the other two sides.
 7.2 The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.

- 7.3** If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal and the larger angle lies opposite the longer side.
- 7.4** If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal and the longer side lies opposite the larger angle.

VOCABULARY

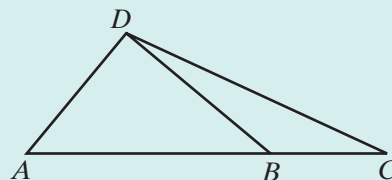
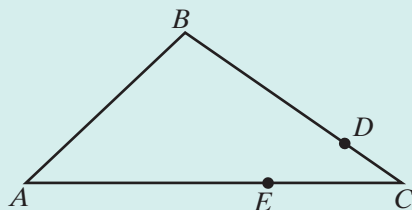
- 7-1** Transitive property of inequality • Trichotomy postulate
- 7-4** Triangle inequality theorem
- 7-5** Exterior angle of a polygon • Adjacent interior angle • Nonadjacent interior angle • Remote interior angle • Exterior angle inequality theorem

REVIEW EXERCISES

In 1–8, state a definition, postulate, or theorem that justifies each of the following statements about the triangles in the figure.

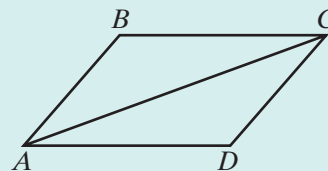
- $AC > BC$
- If $DA < DB$ and $DB < DC$, then $DA < DC$.
- $m\angle DBC > m\angle A$
- If $m\angle C > m\angle CDB$, then $DB > BC$.
- If $DA < DB$, then $DA + AC < DB + AC$.
- $DA + AC > DC$
- If $m\angle A > m\angle C$, then $DC > DA$.
- $m\angle ADC > m\angle ADB$
- Given: \overline{AEC} , \overline{BDC} , $AE > BD$,
and $EC > DC$

Prove: $m\angle B > m\angle A$

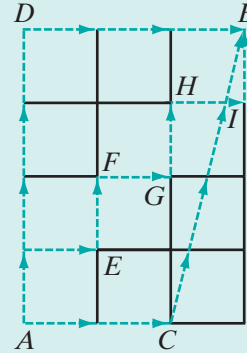


- Given: $\triangle ABC \cong \triangle CDA$,
 $AD > DC$

Prove: **a.** $m\angle ACD > m\angle CAD$
b. \overline{AC} does not bisect $\angle A$.



11. In isosceles triangle ABC , $\overline{CA} \cong \overline{CB}$. If D is a point on \overline{AC} between A and C , prove that $DB > DA$.
12. In isosceles triangle RST , $RS = ST$. Prove that $\angle SRP$, the exterior angle at R , is congruent to $\angle STQ$, the exterior angle at T .
13. Point B is 4 blocks north and 3 blocks east of A . All streets run north and south or east and west except a street that slants from C to B . Of the three paths from A and B that are marked:
- Which path is shortest? Justify your answer.
 - Which path is longest? Justify your answer.



Exploration

The Hinge Theorem states: If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first triangle is greater than the third side of the second triangle.

- With a partner or in a small group, prove the Hinge Theorem.
 - Compare your proof with the proofs of the other groups. Were different diagrams used? Were different approaches used? Were these approaches valid?

The converse of the Hinge Theorem states: If the two sides of one triangle are congruent to two sides of another triangle, and the third side of the first triangle is greater than the third side of the second triangle, then the included angle of the first triangle is larger than the included angle of the second triangle.

- With a partner or in a small group, prove the converse of the Hinge Theorem.
 - Compare your proof with the proofs of the other groups. Were different diagrams used? Were different approaches used? Were these approaches valid?

CUMULATIVE REVIEW

CHAPTERS 1-7

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

- The solution set of the equation $2x - 3.5 = 5x - 18.2$ is
 (1) 49 (2) -49 (3) 4.9 (4) -4.9
- Which of the following is an example of the transitive property of inequality?
 (1) If $a > b$, then $b < a$. (3) If $a > b$ and $c > 0$, then $ac > bc$.
 (2) If $a > b$, then $a + c > b + c$. (4) If $a > b$ and $b > c$, then $a > c$.
- Point M is the midpoint of \overline{ABMC} . Which of the following is *not* true?
 (1) $AM = MC$ (2) $AB < MC$ (3) $AM > BC$ (4) $BM < MC$
- The degree measure of the larger of two complementary angles is 30 more than one-half the measure of the smaller. The degree measure of the smaller is
 (1) 40 (2) 50 (3) 80 (4) 100
- Which of the following could be the measures of the sides of a triangle?
 (1) 2, 2, 4 (2) 1, 3, 5 (3) 7, 12, 20 (4) 6, 7, 12
- Which of the following statements is true for all values of x ?
 (1) $x = 5$ and $x \neq 5$ (3) If $x > 5$, then $x > 3$.
 (2) $x < 5$ or $x > 5$ (4) If $x > 3$, then $x > 5$.
- In $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \cong \overline{DE}$, and $\angle A \cong \angle D$. In order to prove $\triangle ABC \cong \triangle DEF$ using ASA, we need to prove that
 (1) $\angle B \cong \angle E$ (3) $\overline{BC} \cong \overline{EF}$
 (2) $\angle C \cong \angle F$ (4) $\overline{AC} \cong \overline{DF}$
- Under a reflection in the y -axis, the image of $(-2, 5)$ is
 (1) $(2, 5)$ (2) $(2, -5)$ (3) $(-2, -5)$ (4) $(5, -2)$
- Under an opposite isometry, the property that is changed is
 (1) distance (3) collinearity
 (2) angle measure (4) orientation
- Points P and Q lie on the perpendicular bisector of \overline{AB} . Which of the following statements must be true?
 (1) \overline{AB} is the perpendicular bisector of \overline{PQ} .
 (2) $PA = PB$ and $QA = QB$.
 (3) $PA = QA$ and $PB = QB$.
 (4) P is the midpoint of \overline{AB} or Q is the midpoint of \overline{AB} .

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Each of the following statements is true.

If the snow continues to fall, our meeting will be cancelled.

Our meeting is not cancelled.

Can you conclude that snow does not continue to fall? List the logic principles needed to justify your answer.

12. The vertices of $\triangle ABC$ are $A(0, 3)$, $B(4, 3)$, and $C(3, 5)$. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under the composition $r_{y=x} \circ T_{-4, -5}$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Given: \overline{PR} bisects \overline{ARB} but \overline{PR} is not perpendicular to \overline{ARB} .

Prove: $AP \neq BP$

14. Given: In quadrilateral $ABCD$, \overrightarrow{AC} bisects $\angle DAB$ and \overrightarrow{CA} bisects $\angle DCB$.

Prove: $\angle B \cong \angle D$

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The intersection of \overleftrightarrow{PQ} and \overleftrightarrow{RS} is T . If $m\angle PTR = x$, $m\angle QTS = y$, and $m\angle RTQ = 2x + y$, find the measures of $\angle PTR$, $\angle QTS$, $\angle RTQ$, and $\angle PTS$.
16. In $\triangle ABC$, $m\angle A < m\angle B$ and $\angle DCB$ is an exterior angle at C . The measure of $\angle BCA = 6x + 8$, and the measure of $\angle DCB = 4x + 12$.
- Find $m\angle BCA$ and $m\angle DCB$.
 - List the interior angles of the triangle in order, starting with the smallest.
 - List the sides of the triangles in order starting with the smallest.