CHAPTER

13

QUADRATIC RELATIONS AND FUNCTIONS

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When a baseball is hit, its path is not a straight line. The baseball rises to a maximum height and then falls, following a curved path throughout its flight. The maximum height to which it rises is determined by the force with which the ball was hit and the angle at which it was hit. The height of the ball at any time can be found by using an equation, as can the maximum height to which the ball rises and the distance between the batter and the point where the ball hits the ground.

In this chapter we will study the quadratic equation that models the path of a baseball as well as functions and relations that are not linear.

13-1 SOLVING QUADRATIC EQUATIONS

The equation $x^2 - 3x - 10 = 0$ is a polynomial equation in one variable. This equation is of degree two, or second degree, because the greatest exponent of the variable x is 2. The equation is in **standard form** because all terms are collected in descending order of exponents in one side, and the other side is 0.

A **polynomial equation of degree two** is also called a **quadratic equation**. The standard form of a quadratic equation in one variable is

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

To write an equation such as $x(x - 4) = 5$ in standard form, rewrite the left side without parentheses and add -5 to both sides to make the right side 0.

$$\begin{aligned} x(x-4) &= 5 \\ x^2 - 4x &= 5 \\ x^2 - 4x - 5 &= 0 \end{aligned}$$

Solving a Quadratic Equation by Factoring

When 0 is the product of two or more factors, at least one of the factors must be equal to 0. This is illustrated by the following examples:

$$5 \times 0 = 0 \qquad (-2) \times 0 = 0 \qquad \tfrac{1}{2} \times 0 = 0$$
$$0 \times 7 = 0 \qquad 0 \times (-3) = 0 \qquad 0 \times 0 = 0$$

In general:

► **When a and b are real numbers, $ab = 0$ if and only if $a = 0$ or $b = 0$.**

This principle is used to solve quadratic equations. For example, to solve the quadratic equation $x^2 - 3x + 2 = 0$, we can write the left side as $(x - 2)(x - 1)$. The factors $(x - 2)$ and $(x - 1)$ represent real numbers whose product is 0. The equation will be true if the first factor is 0, that is, if $(x - 2) = 0$ or if the second factor is 0, that is, if $(x - 1) = 0$.

$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0$$

$$\begin{array}{rcr} x - 2 &=& 0 \\ \underline{\;\;+2} && \underline{+2} \\ x &=& 2 \end{array} \qquad \Bigg| \qquad \begin{array}{rcr} x - 1 &=& 0 \\ \underline{\;\;+1} && \underline{+1} \\ x &=& 1 \end{array}$$

A check will show that both 2 and 1 are values of x for which the equation is true.

Check for $x = 2$:

$$x^2 - 3x + 2 = 0$$
$$(2)^2 - 3(2) + 2 \stackrel{?}{=} 0$$
$$4 - 6 + 2 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

Check for $x = 1$:

$$x^2 - 3x + 2 = 0$$
$$(1)^2 - 3(1) + 2 \stackrel{?}{=} 0$$
$$1 - 3 + 2 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

Since both 2 and 1 satisfy the equation $x^2 - 3x + 2 = 0$, the solution set of this equation is {2, 1}. The **roots of the equation**, that is, the values of the variable that make the equation true, are 2 and 1.

Note that the factors of the trinomial $x^2 - 3x + 2$ are $(x - 2)$ and $(x - 1)$. If the trinomial $x^2 - 3x + 2$ is set equal to zero, then an equation is formed and this equation has a solution set, {2, 1}.

Procedure

To solve a quadratic equation by factoring:

1. If necessary, transform the equation into standard form.
2. Factor the quadratic expression.
3. Set each factor containing the variable equal to 0.
4. Solve each of the resulting equations.
5. Check by substituting each value of the variable in the original equation.

► **The real number k is a root of $ax^2 + bx + c = 0$ if and only if $(x - k)$ is a factor of $ax^2 + bx + c$.**

EXAMPLE 1

Solve and check: $x^2 - 7x = -10$

Solution *How to Proceed*

(1) Write the equation in standard form: $x^2 - 7x = -10$

$x^2 - 7x + 10 = 0$

(2) Factor the quadratic expression: $(x - 2)(x - 5) = 0$

(3) Let each factor equal 0: $x - 2 = 0$ | $x - 5 = 0$

(4) Solve each equation: $x = 2$ | $x = 5$

(5) Check both values in the original equation:

Check for $x = 2$:

$$x^2 - 7x = -10$$
$$(2)^2 - 7(2) \stackrel{?}{=} -10$$
$$4 - 14 \stackrel{?}{=} -10$$
$$-10 = -10 \checkmark$$

Check for $x = 5$:

$$x^2 - 7x = -10$$
$$(5)^2 - 7(5) \stackrel{?}{=} -10$$
$$25 - 35 \stackrel{?}{=} -10$$
$$-10 = -10 \checkmark$$

Answer $x = 2$ or $x = 5$; the solution set is $\{2, 5\}$.

EXAMPLE 2

List the members of the solution set of $2x^2 = 3x$.

Solution

How to Proceed

(1) Write the equation in standard form: $2x^2 = 3x$

$$2x^2 - 3x = 0$$

(2) Factor the quadratic expression: $x(2x - 3) = 0$

(3) Let each factor equal 0: $x = 0$ | $2x - 3 = 0$

(4) Solve each equation: $2x = 3$

$$x = \tfrac{3}{2}$$

(5) Check both values in the original equation.

Check for $x = 0$:

$$2x^2 = 3x$$
$$2(0)^2 \stackrel{?}{=} 3(0)$$
$$0 = 0 \checkmark$$

Check for $x = \frac{3}{2}$:

$$2x^2 = 3x$$
$$2\left(\tfrac{3}{2}\right)^2 \stackrel{?}{=} 3\left(\tfrac{3}{2}\right)$$
$$2\left(\tfrac{9}{4}\right) \stackrel{?}{=} 3\left(\tfrac{3}{2}\right)$$
$$\tfrac{9}{2} = \tfrac{9}{2} \checkmark$$

Answer: The solution set is $\left\{0, \frac{3}{2}\right\}$.

Note: We never divide both sides of an equation by an expression containing a variable. If we had divided $2x^2 = 3x$ by x, we would have obtained the equation $2x = 3$, whose solution is $\frac{3}{2}$ but would have lost the solution $x = 0$.

EXAMPLE 3

Find the solution set of the equation $x(x - 8) = 2x - 25$.

Solution *How to Proceed*

(1) Use the distributive property on the left side of the equation:
$$x(x - 8) = 2x - 25$$
$$x^2 - 8x = 2x - 25$$

(2) Write the equation in standard form:
$$x^2 - 10x + 25 = 0$$

(3) Factor the quadratic expression:
$$(x - 5)(x - 5) = 0$$

(4) Let each factor equal 0:
$$x - 5 = 0 \quad \Big| \quad x - 5 = 0$$

(5) Solve each equation:
$$x = 5 \quad \Big| \quad x = 5$$

(6) Check the value in the original equation:
$$x(x - 8) = 2x - 25$$
$$5(5 - 8) \stackrel{?}{=} 2(5) - 25$$
$$-15 \stackrel{?}{=} 10 - 25$$
$$-15 = -15 \checkmark$$

Answer $x = 5$; the solution set is {5}.

Every quadratic equation has two roots, but as Example 3 shows, the two roots are sometimes the same number. Such a root, called a **double root**, is written only once in the solution set.

EXAMPLE 4

The height h of a ball thrown into the air with an initial vertical velocity of 24 feet per second from a height of 6 feet above the ground is given by the equation $h = -16t^2 + 24t + 6$ where t is the time, in seconds, that the ball has been in the air. After how many seconds is the ball at a height of 14 feet?

Solution (1) In the equation, let $h = 14$:
$$h = -16t^2 + 24t + 6$$
$$14 = -16t^2 + 24t + 6$$

(2) Write the equation in standard form:
$$0 = -16t^2 + 24t - 8$$

(3) Factor the quadratic expression:
$$0 = -8(2t^2 - 3t + 1)$$
$$0 = -8(2t - 1)(t - 1)$$

(4) Solve for t:
$$2t - 1 = 0 \quad \Big| \quad t - 1 = 0$$
$$2t = 1 \quad \Big| \quad t = 1$$
$$t = \tfrac{1}{2}$$

Answer The ball is at a height of 14 feet after $\frac{1}{2}$ second as it rises and after 1 second as it falls.

EXAMPLE 5

The area of a circle is equal to 3 times its circumference. What is the radius of the circle?

Solution *How to Proceed*

(1) Write an equation from the given information: $\pi r^2 = 3(2\pi r)$

(2) Set the equation in standard form: $\pi r^2 - 6\pi r = 0$

(3) Factor the quadratic expression: $\pi r(r - 6) = 0$

(4) Solve for r: $\pi r = 0$ | $r - 6 = 0$

(5) Reject the zero value. Use the positive value to write the answer. $r = 0$ | $r = 6$

Answer The radius of the circle is 6 units.

EXERCISES

Writing About Mathematics

1. Can the equation $x^2 = 9$ be solved by factoring? Explain your answer.

2. In Example 4, the trinomial was written as the product of three factors. Only two of these factors were set equal to 0. Explain why the third factor was not used to find a solution of the equation.

Developing Skills

In 3–38, solve each equation and check.

3. $x^2 - 3x + 2 = 0$ **4.** $z^2 - 5z + 4 = 0$ **5.** $x^2 - 8x + 16 = 0$

6. $r^2 - 12r + 35 = 0$ **7.** $c^2 + 6c + 5 = 0$ **8.** $m^2 + 10m + 9 = 0$

9. $x^2 + 2x + 1 = 0$ **10.** $y^2 + 11y + 24 = 0$ **11.** $x^2 - 4x - 5 = 0$

12. $x^2 + x - 6 = 0$ **13.** $x^2 + 2x - 15 = 0$ **14.** $r^2 - r - 72 = 0$

15. $x^2 - x - 12 = 0$ **16.** $x^2 - 49 = 0$ **17.** $z^2 - 4 = 0$

18. $m^2 - 64 = 0$ **19.** $3x^2 - 12 = 0$ **20.** $d^2 - 2d = 0$

21. $s^2 - s = 0$ **22.** $x^2 + 3x = 0$ **23.** $z^2 + 8z = 0$

24. $x^2 - x = 6$ **25.** $y^2 - 3y = 28$ **26.** $c^2 - 8c = -15$

27. $r^2 = 4$ **28.** $x^2 = 121$ **29.** $y^2 = 6y$

30. $s^2 = -4s$ **31.** $y^2 = 8y + 20$ **32.** $x^2 = 9x - 20$

33. $30 + x = x^2$ **34.** $x^2 + 3x - 4 = 50$ **35.** $2x^2 + 7 = 5 - 5x$

36. $x(x - 2) = 35$ **37.** $y(y - 3) = 4$ **38.** $x(x + 3) = 40$

In 39–44, solve each equation and check.

39. $\frac{x + 2}{2} = \frac{12}{x}$

40. $\frac{y + 3}{3} = \frac{6}{y}$

41. $\frac{x}{3} = \frac{12}{x}$

42. $\frac{x + 4}{-1} = \frac{4}{x}$

43. $\frac{-4x}{x - 3} = \frac{x - 1}{2}$

44. $\frac{2x - 2}{x + 3} = \frac{x - 1}{x - 2}$

Applying Skills

45. The height h of a ball thrown into the air with an initial vertical velocity of 48 feet per second from a height of 5 feet above the ground is given by the equation

$$h = -16t^2 + 48t + 5$$

where t is the time in seconds that the ball has been in the air. After how many seconds is the ball at a height of 37 feet?

46. A batter hit a baseball at a height of 4 feet with a force that gave the ball an initial vertical velocity of 64 feet per second. The equation

$$h = -16t^2 + 64t + 4$$

gives the height h of the baseball t seconds after the ball was hit. If the ball was caught at a height of 4 feet, how long after the batter hit the ball was the ball caught?

47. The length of a rectangle is 12 feet more than twice the width. The area of the rectangle is 320 square feet.

a. Write an equation that can be used to find the length and width of the rectangle.

b. What are the dimensions of the rectangle?

48. A small park is enclosed by four streets, two of which are parallel. The park is in the shape of a trapezoid. The perpendicular distance between the parallel streets is the height of the trapezoid. The portions of the parallel streets that border the park are the bases of the trapezoid. The height of the trapezoid is equal to the length of one of the bases and 20 feet longer than the other base. The area of the park is 9,000 square feet.

a. Write an equation that can be used to find the height of the trapezoid.

b. What is the perpendicular distance between the two parallel streets?

49. At a kennel, each dog run is a rectangle whose length is 4 feet more than twice the width. Each run encloses 240 square feet. What are the dimensions of the runs?

50. One leg of a right triangle is 14 centimeters longer than the other leg. The length of the hypotenuse is 26 centimeters. What are the lengths of the legs?

13-2 THE GRAPH OF A QUADRATIC FUNCTION

A batter hits a baseball at a height of 3 feet off the ground, with an initial vertical velocity of 72 feet per second. The height, y, of the baseball can be found using the equation $y = -16x^2 + 72x + 3$ when x represents the number of seconds from the time the ball was hit. The graph of the equation—and, incidentally, the actual path of the ball—is a curve called a **parabola**. The special properties of parabolas are discussed in this section.

An equation of the form $y = ax^2 + bx + c$ $(a \neq 0)$ is called a **second-degree polynomial function** or a **quadratic function**. It is a function because for every ordered pair in its solution set, each different value of x is paired with one and only one value of y. The graph of any quadratic function is a parabola.

Because the graph of a quadratic function is nonlinear, a larger number of points are needed to draw the graph than are needed to draw the graph of a linear function. The graphs of two equations of the form $y = ax^2 + bx + c$, one that has a positive coefficient of x^2 $(a > 0)$ and the other a negative coefficient of x^2 $(a < 0)$, are shown below.

CASE 1 *The graph of $ax^2 + bx + c$ where $a > 0$.*

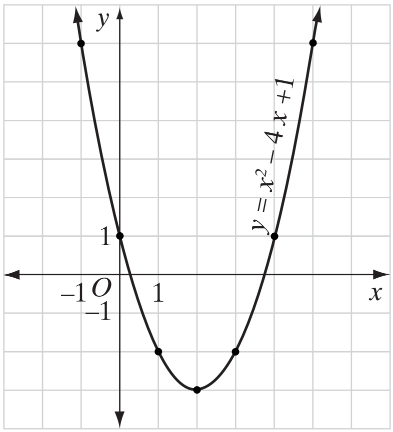
Graph the quadratic function $y = x^2 - 4x + 1$ for integral values of x from -1 to 5 inclusive:

(1) Make a table using integral values of x from -1 to 5.

(2) Plot the points associated with each ordered pair (x, y).

(3) Draw a smooth curve through the points.

x	$x^2 - 4x + 1$	y
-1	$1 + 4 + 1$	6
0	$0 - 0 + 1$	1
1	$1 - 4 + 1$	-2
2	$4 - 8 + 1$	-3
3	$9 - 12 + 1$	-2
4	$16 - 16 + 1$	1
5	$25 - 20 + 1$	6



The values of x that were chosen to draw this graph are not a random set of numbers. These numbers were chosen to produce the pattern of y-values shown in the table. Notice that as x increases from -1 to 2, y decreases from 6 to -3. Then the graph reverses and as x continues to increase from 2 to 5, y increases from -3 to 6. The smallest value of y occurs at the point $(2, -3)$. The point is called the **minimum** because its y-value, -3, is the smallest value of y for the equation. The minimum point is also called the **turning point** or **vertex** of the parabola.

The graph is symmetric with respect to the vertical line, called the **axis of symmetry of the parabola**. The axis of symmetry of the parabola is determined by the formula

$$x = \frac{-b}{2a}$$

where a and b are the coefficients x^2 and x, respectively, from the standard form of the quadratic equation.

For the function $y = x^2 - 4x + 1$, the equation of the vertical line of symmetry is $x = \frac{-(-4)}{2(1)}$ or $x = 2$. Every point on the parabola to the left of $x = 2$ matches a point on the parabola to the right of $x = 2$, and vice versa.

This example illustrates the following properties of the graph of the quadratic equation $y = x^2 - 4x + 1$:

1. The graph of the equation is a parabola.
2. The parabola is symmetric with respect to the vertical line $x = 2$.
3. The parabola opens upward and has a minimum point at $(2, -3)$.
4. The equation defines a function. For every x-value there is one and only one y-value.
5. The constant term, 1, is the y-intercept. The y-intercept is the value of y when x is 0.

CASE 2 *The graph of* $ax^2 + bx + c$ *where* $a < 0$.

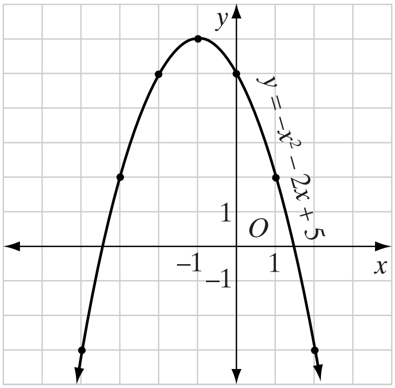
Graph the quadratic function $y = -x^2 - 2x + 5$ using integral values of x from -4 to 2 inclusive:

(1) Make a table using integral values of x from -4 to 2.

(2) Plot the points associated with each ordered pair (x, y).

(3) Draw a smooth curve through the points.

x	$-x^2 - 2x + 5$	y
-4	$-16 + 8 + 5$	-3
-3	$-9 + 6 + 5$	2
-2	$-4 + 4 + 5$	5
-1	$-1 + 2 + 5$	6
0	$0 - 0 + 5$	5
1	$-1 - 2 + 5$	2
2	$-4 - 4 + 5$	-3



Again, the values of x that were chosen to produce the pattern of y-values shown in the chart. Notice that as x increases from -4 to -1, y increases from

-3 to 6. Then the graph reverses, and as x continues to increase from -1 to 2, y decreases from 6 to -3. The largest value of y occurs at the point $(-1, 6)$. This point is called the **maximum** because its y-value, 6, is the largest value of y for the equation. In this case, the maximum point is the turning point or vertex of the parabola.

The graph is symmetric with respect to the vertical line whose equation is $x = -1$. As shown in Case 1, this value of x is again given by the formula

$$x = \frac{-b}{2a}$$

where a and b are the coefficients of x^2 and x, respectively. For the function $y = -x^2 - 2x + 5$, the equation of the axis of symmetry is $x = \frac{-(-2)}{2(-1)}$ or $x = -1$.

This example illustrates the following properties of the graph of the quadratic equation $y = -x^2 - 2x + 5$:

1. The graph of the equation is a parabola.
2. The parabola is symmetric with respect to the vertical line $x = -1$.
3. The parabola opens downward and has a maximum point at $(-1, 6)$
4. The equation defines a function.
5. The constant term, 5, is the y-intercept.

When the equation of a parabola is written in the form $y = ax^2 + bx + c = 0$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$ and the x-coordinate of the turning point is $\frac{-b}{2a}$. This can be used to find a convenient set of values of x to be used when drawing a parabola. Use $\frac{-b}{2a}$ as a middle value of x with three values that are smaller and three that are larger.

KEEP IN MIND

1. The graph of $y = ax^2 + bx + c$, with $a \neq 0$, is a parabola.
2. The axis of symmetry of the parabola is a vertical line whose equation is $x = \frac{-b}{2a}$.
3. A parabola has a turning point on the axis of symmetry. The x-coordinate of the turning point is $\frac{-b}{2a}$. The y-coordinate of the turning point is found by substituting $\frac{-b}{2a}$ into the equation of the parabola. The turning point is the vertex of the parabola.
4. If a is positive, the parabola opens upward and the turning point is a minimum. The minimum value of y for the parabola is the y-coordinate of the turning point.
5. If a is negative, the parabola opens downward and the turning point is a maximum. The maximum value of y for the parabola is the y-coordinate of the turning point.

EXAMPLE 1

a. Write the equation of the axis of symmetry of $y = x^2 - 3$.

b. Graph the function.

c. Does the function have a maximum or a minimum?

d. What is the maximum or minimum value of the function?

e. Write the coordinates of the vertex.

Solution **a.** In this equation, $a = 1$. Since there is no x term in the equation, the equation can be written as $y = x^2 + 0x - 3$ with $b = 0$. The equation of the axis of symmetry is

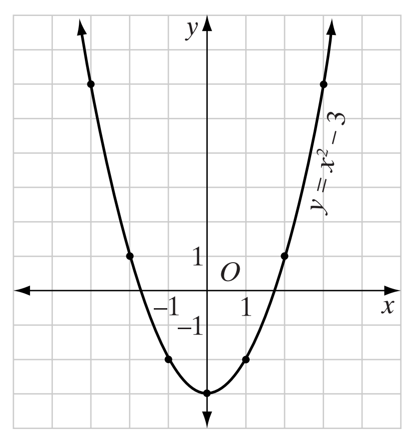
$$x = \frac{-b}{2a} = \frac{-(0)}{2(1)} = 0 \text{ or } x = 0.$$

b. (1) Since the vertex of the parabola is on the axis of symmetry, the x-coordinate of the vertex is 0. Use three values of x that are less than 0 and three values of x that are greater than 0. Make a table using integral values of x from -3 to 3.

(2) Plot the points associated with each ordered pair (x, y).

(3) Draw a smooth curve through the points to draw a parabola.

x	$x^2 - 3$	y
-3	$(-3)^2 - 3$	6
-2	$(-2)^2 - 3$	1
-1	$(-1)^2 - 3$	-2
0	$(0)^2 - 3$	-3
1	$(1)^2 - 3$	-2
2	$(2)^2 - 3$	1
3	$(3)^2 - 3$	6



c. Since $a = 1 > 0$, the function has a minimum.

d. The minimum value of the function is the y-coordinate of the vertex, -3, which can be read from the table of values.

e. Since the vertex is the turning point of this parabola, the coordinates of the vertex are $(0, -3)$.

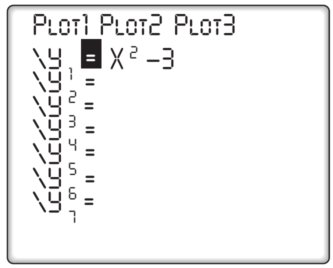
Calculator Solution

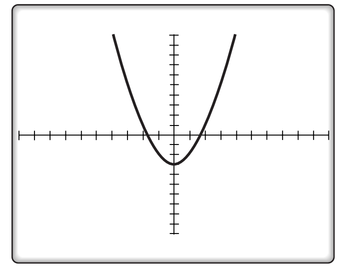
a. Determine the equation of the axis of symmetry as before.

b. Enter the equation in the Y= list of functions and graph the function. Clear any equations already in the list.

ENTER: [Y=] [X,T,Θ,*n*] [x^2] [−] 3 [ZOOM] [6]

DISPLAY:



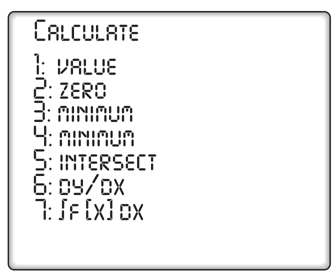


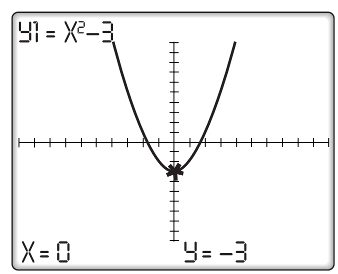
c. The graph shows that the function has a minimum.

d. Since the minimum of the function occurs at the vertex, use value ([2nd] [CALC] [1]) from the CALC menu to evaluate the function at $x = 0$, the x-coordinate of the vertex:

ENTER: [2nd] [CALC] [1] 0 [ENTER]

DISPLAY:





The calculator displays the minimum value, −3.

e. The coordinates of the vertex are (0, −3).

Answers

a. The axis of symmetry is the y-axis. The equation is $x = 0$.

b. Graph

c. The function has a minimum.

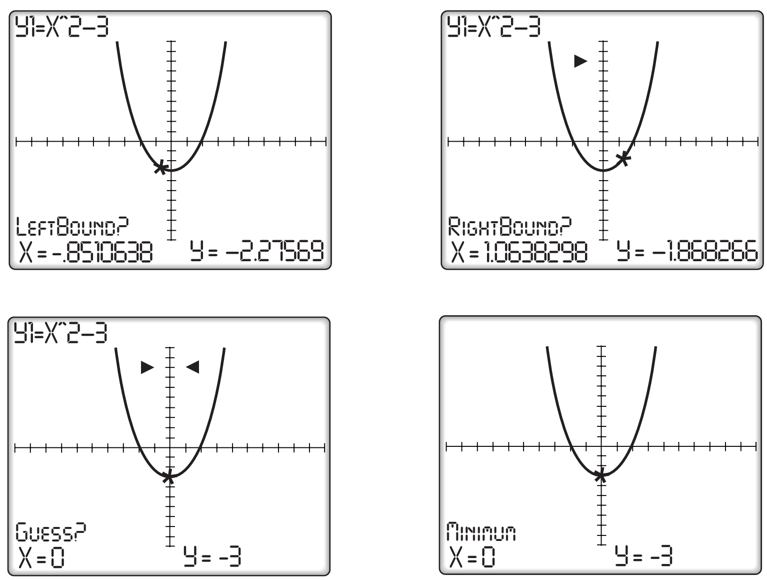
d. The minimum value is −3.

e. The vertex is (0, −3).

When the coordinates of the turning point are rational numbers, we can use minimum (2nd CALC 3) or maximum (2nd CALC 4) from the CALC menu to find the vertex. In Example 1, since the turning point is a minimum, use minimum:

ENTER: 2nd CALC 3

When the calculator asks "LeftBound?" move the cursor to any point to the left of the vertex using the left or right arrow keys, and then press enter. When the calculator asks "RightBound?" move the cursor to the right of the vertex, and then press enter. When the calculator asks "Guess?" move the cursor near the vertex and then press enter. The calculator displays the coordinates of the vertex at the bottom of the screen.



EXAMPLE 2

Sketch the graph of the function $y = x^2 - 7x + 10$.

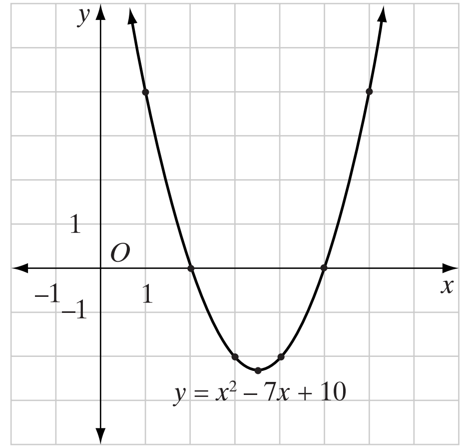
Solution (1) Find the equation of the axis of symmetry and the x-coordinate of the vertex:

$$x = \frac{-b}{2a} = \frac{-(-7)}{2(1)} = 3.5$$

(2) Make a table of values using three integral values smaller than and three larger than 3.5.

(3) Plot the points whose coordinates are given in the table and draw a smooth curve through them.

x	$x^2 - 7x + 10$	y
1	$1 - 7 + 10$	4
2	$4 - 14 + 10$	0
3	$9 - 21 + 10$	-2
3.5	$12.25 - 24.5 + 10$	-2.25
4	$16 - 28 + 10$	-2
5	$25 - 35 + 10$	0
6	$36 - 42 + 10$	4



Note: The table can also be displayed on the calculator. First enter the equation into Y_1.

ENTER: [Y=] [X,T,Θ,n] [x²] [−] 7 [X,T,Θ,n] [+] 10

Then enter the starting value and the interval between the x values. We will use 1 as the starting value and 0.5 as the interval in order to include 3.5, the x-value of the vertex.

ENTER: [2nd] [TBLSET] 1 [ENTER] .5 [ENTER]

Before creating the table, make sure that "Indpnt:" and "Depend:" are set to "auto." If they are not, press [ENTER] [▼] [ENTER]. Finally, press [2nd] [TABLE] to create the table. Scroll up and down to view the values of x and y.

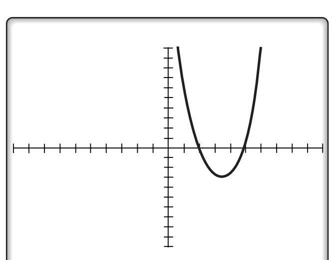
X	Y1	
1	4	
1.5	1.75	
2	0	
2.5	-1.25	
3	-2	
3.5	-2.25	
4	-2	

X=1

Calculator Solution Enter the equation in the Y= menu and sketch the graph of the function in the standard window.

ENTER: [Y=] [X,T,Θ,n] [x²] [−] 7 [X,T,Θ,n] [+] 10 [ZOOM] [6]

DISPLAY:



EXAMPLE 3

The perimeter of a rectangle is 12. Let x represent the measure of one side of the rectangle and y represent the area.

a. Write an equation for the area of the rectangle in terms of x.

b. Draw the graph of the equation written in **a**.

c. What is the maximum area of the rectangle?

Solution **a.** Let x be the measure of the length of the rectangle. Use the formula for perimeter to express the measure of the width in terms of x:

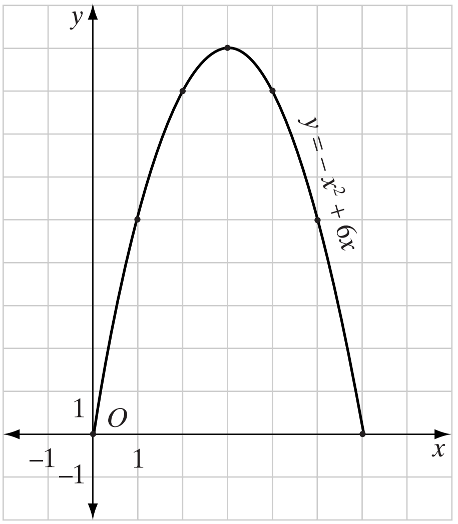
$$P = 2l + 2w$$
$$12 = 2x + 2w$$
$$6 = x + w$$
$$w = 6 - x$$

Write the formula for area in terms of x and y:

$$A = lw$$
$$y = x(6 - x)$$
$$y = 6x - x^2$$
$$y = -x^2 + 6x$$

b. The equation of the axis of symmetry is $x = \frac{-6}{2(-1)}$ or $x = 3$. Make a table of values using values of x on each side of 3.

x	$-x^2 + 6x$	y
0	$-0 + 0$	0
1	$-1 + 6$	5
2	$-4 + 12$	8
3	$-9 + 18$	9
4	$-16 + 24$	8
5	$-25 + 30$	5
6	$-36 + 36$	0



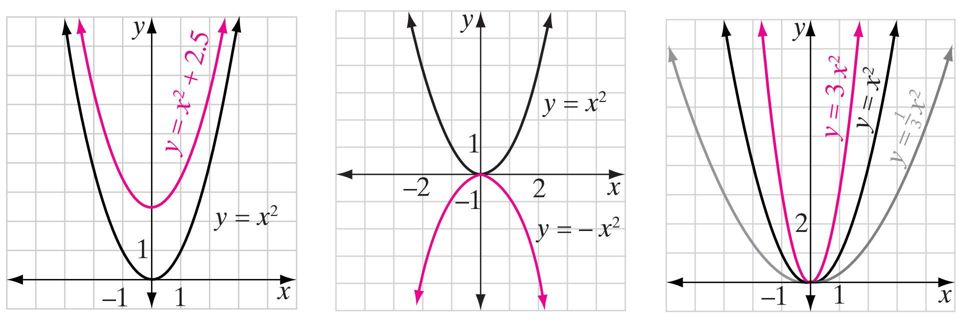
c. The maximum value of the area, y, is 9.

Note: The graph shows all possible values of x and y. Since both the measure of a side of the rectangle, x, and the area of the rectangle, y, must be positive, $0 < x < 6$ and $0 < y \leq 9$. Since (2, 8) is a point on the graph, one possible rectangle has dimensions 2 by $(6 - 2)$ or 2 by 4 and an area of 8. The rectangle with maximum area, 9, has dimensions 3 by $(6 - 3)$ or 3 by 3, a square.

Translating, Reflecting, and Scaling Graphs of Quadratic Functions

Just as linear and absolute value functions can be translated, reflected, or scaled, graphs of quadratic functions can also be manipulated by working with the graph of the quadratic function $y = x^2$.

For instance, the graph of $y = x^2 + 2.5$ is the graph of $y = x^2$ shifted 2.5 units up. The graph of $y = -x^2$ is the graph of $y = x^2$ reflected in the x-axis. The graph of $y = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, while the graph of $y = \frac{1}{3}x^2$ is the graph of $y = x^2$ compressed vertically by a factor of $\frac{1}{3}$.



Translation Rules for Quadratic Functions

If c is *positive*:

- **The graph of $y = x^2 + c$ is the graph of $y = x^2$ shifted c units up.**
- **The graph of $y = x^2 - c$ is the graph of $y = x^2$ shifted c units down.**
- **The graph of $y = (x + c)^2$ is the graph of $y = x^2$ shifted c units to the left.**
- **The graph of $y = (x - c)^2$ is the graph of $y = x^2$ shifted c units to the right.**

Reflection Rule for Quadratic Functions

- **The graph of $y = -x^2$ is the graph of $y = x^2$ reflected in the x-axis.**

Scaling Rules for Quadratic Functions

- **When $c > 1$, the graph of $y = cx^2$ is the graph of $y = x^2$ stretched vertically by a factor of c.**
- **When $0 < c < 1$, the graph of $y = cx^2$ is the graph of $y = x^2$ compressed vertically by a factor of c.**

EXAMPLE 4

In **a–e**, write an equation for the resulting function if the graph of $y = x^2$ is:

a. shifted 5 units down and 1.5 units to the left

b. stretched vertically by a factor of 4 and shifted 2 units down

c. compressed vertically by a factor of $\frac{1}{6}$ and reflected in the x-axis

d. reflected in the x-axis, shifted 2 units up, and shifted 2 units to the right

Solution **a.** $y = (x + 1.5)^2 - 5$ *Answer*

b. First, stretch vertically by a factor of 4: $y = 4x^2$

Then, translate the resulting function 2 units down: $y = 4x^2 - 2$ *Answer*

c. First, compress vertically by a factor of $\frac{1}{6}$: $y = \frac{1}{6}x^2$

Then, reflect in the x-axis: $y = -\frac{1}{6}x^2$ *Answer*

d. First, reflect in the x-axis: $y = -x^2$

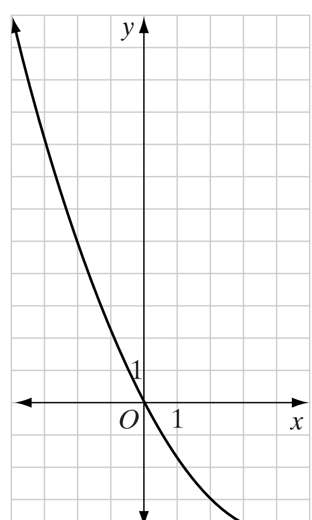
Then, translate the resulting function 2 units up: $y = -x^2 + 2$

Finally, translate the resulting function 2 units to the right: $y = -(x - 2)^2 + 2$ *Answer*

EXERCISES

Writing About Mathematics

1. Penny drew the graph of $h(x) = \frac{1}{4}x^2 - 2x$ from $x = -4$ to $x = 4$. Her graph is shown to the right. Explain why Penny's graph does not look like a parabola.



2. What values of x would you choose to draw the graph of $h(x) = \frac{1}{4}x^2 - 2x$ so that points on both sides of the turning point would be shown on the graph? Explain your answer.

Developing Skills

In 3–14: **a.** Graph each quadratic function on graph paper using the integral values for x indicated in parentheses to prepare the necessary table of values. **b.** Write the equation of the axis of symmetry of the graph. **c.** Write the coordinates of the turning point of the graph.

3. $y = x^2$ $(-3 \leq x \leq 3)$

4. $y = -x^2$ $(-3 \leq x \leq 3)$

5. $y = x^2 + 1$ $(-3 \leq x \leq 3)$

6. $y = x^2 - 1$ $(-3 \leq x \leq 3)$

7. $y = -x^2 + 4$ $(-3 \leq x \leq 3)$

8. $y = x^2 - 2x$ $(-2 \leq x \leq 4)$

9. $y = -x^2 + 2x$ $(-2 \leq x \leq 4)$

10. $y = x^2 - 6x + 8$ $(0 \leq x \leq 6)$

11. $y = x^2 - 4x + 3$ $(-1 \leq x \leq 5)$

12. $y = x^2 - 2x + 1$ $(-2 \leq x \leq 4)$

13. $y = -x^2 - 2x + 3$ $(-4 \leq x \leq 2)$

14. $y = -x^2 + 4x - 3$ $(-1 \leq x \leq 5)$

In 15–20: **a.** Write the equation of the axis of symmetry of the graph of the function. **b.** Find the coordinates of the vertex. **c.** Draw the graph on graph paper or on a calculator, showing at least three points with integral coefficients on each side of the vertex.

15. $y = x^2 - 6x - 1$

16. $y = x^2 - 2x + 8$

17. $y = x^2 + 8x + 12$

18. $y = x^2 + 4x + 3$

19. $y = x^2 - 3x + 7$

20. $y = x^2 + x + 5$

21. Write an equation for the resulting function if the graph of $y = x^2$ is:

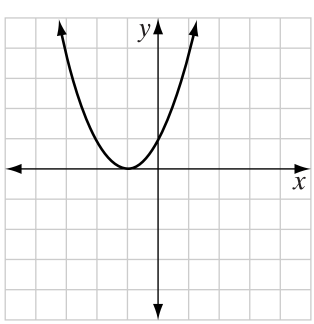
a. reflected in the x-axis and shifted 3 units left.

b. compressed vertically by a factor of $\frac{2}{7}$ and shifted 9 units up.

c. reflected in the x-axis, stretched vertically by a factor of 6, shifted 1 unit down, and shifted 4 units to the right.

In 22–25, each graph is a translation and/or a reflection of the graph of $y = x^2$. For each graph, **a.** determine the vertex and the axis of symmetry, and **b.** write the equation of each graph.

22.



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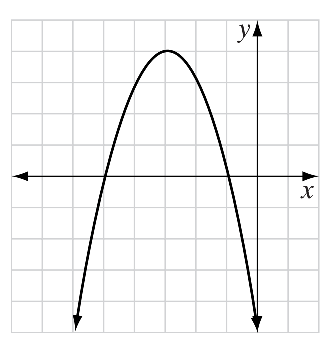

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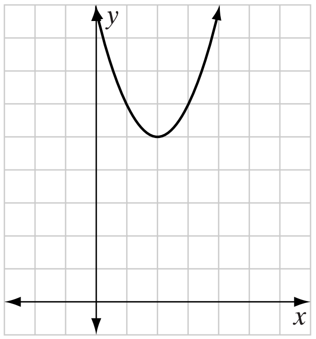

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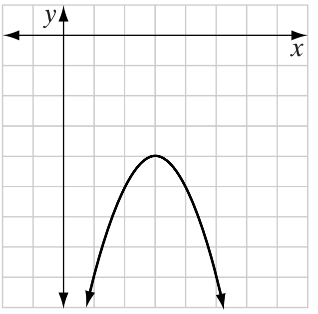
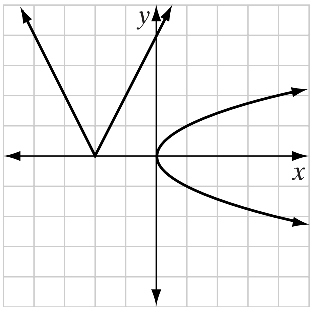
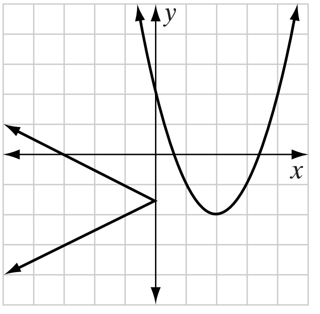


26. Of the graphs below, which is the graph of a quadratic function and the graph of an absolute value function?

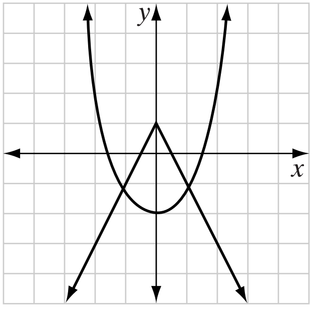
(1)



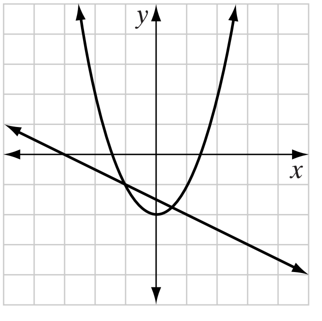
(3)



(2)



(4)



Applying Skills

27. The length of a rectangle is 4 more than its width.

a. If x represents the width of the rectangle, represent the length of the rectangle in terms of x.

b. If y represents the area of the rectangle, write an equation for y in terms of x.

c. Draw the graph of the equation that you wrote in part **b**.

d. Do all of the points on the graph that you drew represent pairs of values for the width and area of the rectangle? Explain your answer.

28. The height of a triangle is 6 less than twice the length of the base.

a. If x represents the length of the base of the triangle, represent the height in terms of x.

b. If y represents the area of the triangle, write an equation for y in terms of x.

c. Draw the graph of the equation that you wrote in part **b**.

d. Do all of the points on the graph that you drew represent pairs of values for the length of the base and area of the triangle? Explain your answer.

29. The perimeter of a rectangle is 20 centimeters. Let x represent the measure of one side of the rectangle and y represent the area of the rectangle.

a. Use the formula for perimeter to express the measure of a second side of the rectangle.

b. Write an equation for the area of the rectangle in terms of x.

c. Draw the graph of the equation written in **b**.

d. What are the dimensions of the rectangle with maximum area?

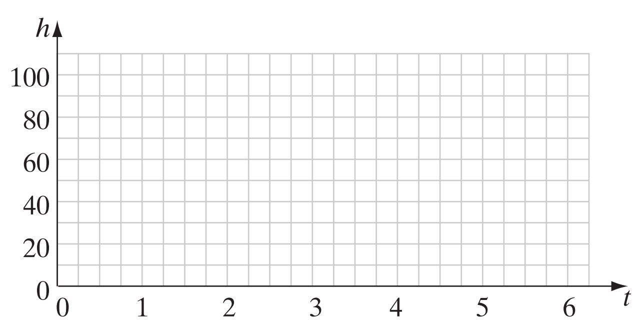
e. What is the maximum area of the rectangle?

f. List four other possible dimensions and areas for the rectangle.

30. A batter hit a baseball at a height 3 feet off the ground, with an initial vertical velocity of 64 feet per second. Let x represent the time in seconds, and y represent the height of the baseball. The height of the ball can be determined over a limited period of time by using the equation $y = -16x^2 + 64x + 3$.

a. Make a table using integral values of x from 0 to 4 to find values of y.

b. Graph the equation. Let one horizontal unit $= \frac{1}{4}$ second, and one vertical unit $= 10$ feet. (See suggested coordinate grid below.)



c. If the ball was caught after 4 seconds, what was its height when it was caught?

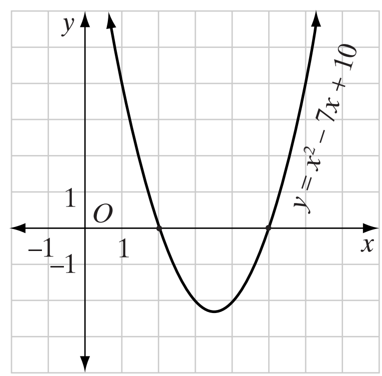
d. From the table and the graph, determine:
(1) the maximum height reached by the baseball;
(2) the time required for the ball to reach this height.

13-3 FINDING ROOTS FROM A GRAPH

In Section 1, you learned to find the solution of an equation of the form $ax^2 + bx + c = 0$ by factoring. In Section 2, you learned to draw the graph of a function of the form $y = ax^2 + bx + c$. How are these similar expressions related? To answer this question, we will consider three possible cases.

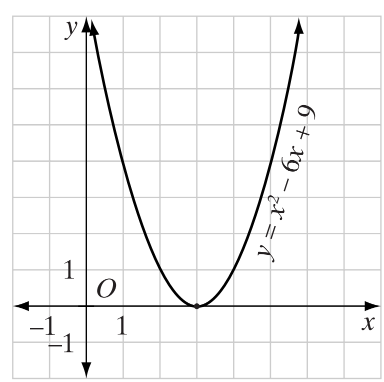
CASE 1 *A quadratic equation can have two distinct real roots.*

The equation $x^2 - 7x + 10 = 0$ has exactly two roots or solutions, 5 and 2, that make the equation true. The function, $y = x^2 - 7x + 10$ has infinitely many pairs of numbers that make the equation true. The graph of this function shown at the right intersects the x-axis in two points, (5, 0) and (2, 0). Since the y-coordinates of these points are 0, the x-coordinates of these points are the roots of the equation $x^2 - 7x + 10 = 0$.



CASE 2 *A quadratic equation can have only one distinct real root.*

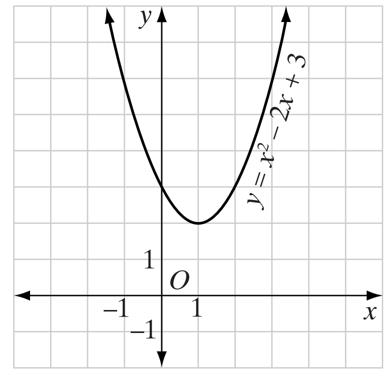
The two roots of the equation $x^2 - 6x + 9 = 0$ are equal. There is only one number, 3, that makes the equation true. The function $y = x^2 - 6x + 9$ has infinitely many pairs of numbers that make the equation true. The graph of this function shown at the right intersects the x-axis in only one point, (3, 0). Since the y-coordinate of this point is 0, the x-coordinate of this point is the root of the equation $x^2 - 6x + 9 = 0$.



Recall from Section 1 that when a quadratic equation has only one distinct root, the root is said to be a *double root*. In other words, when the root of a quadratic equation is a double root, the graph of the corresponding quadratic function intersects the x-axis exactly once.

CASE 3 *A quadratic equation can have no real roots.*

The equation $x^2 - 2x + 3 = 0$ has no real roots. There is no real number that makes the equation true. The function, $y = x^2 - 2x + 3$ has infinitely many pairs of numbers that make the equation true. The graph of this function shown at the right does not intersect the x-axis. There is no point on the graph whose y-coordinate is 0. Since there is no point whose y-coordinate is 0, there are no real numbers that are roots of the equation $x^2 - 2x + 3 = 0$.

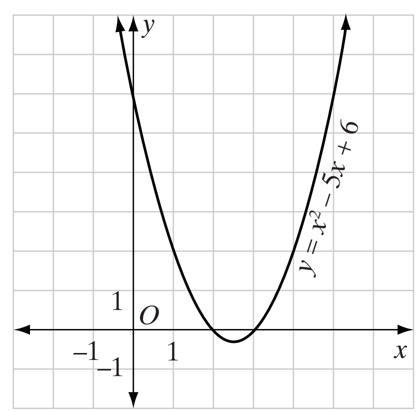


The equation of the x-axis is $y = 0$. The x-coordinates of the points at which the graph of $y = ax^2 + bx + c$ intersects the x-axis are the roots of the equation $ax^2 + bx + c = 0$. The graph of the function $y = ax^2 + bx + c$ can intersect the x-axis in 0, 1, or 2 points, and the equation $ax^2 + bx + c = 0$ can have 0, 1, or 2 real roots.

A real number k is a root of the quadratic equation $ax^2 + bx + c = 0$ if and only if the graph of $y = ax^2 + bx + c$ intersects the x-axis at $(k, 0)$.

EXAMPLE 1

Use the graph of $y = x^2 - 5x + 6$ to find the roots of $x^2 - 5x + 6 = 0$.



Solution The graph intersects the x-axis at $(2, 0)$ and $(3, 0)$.

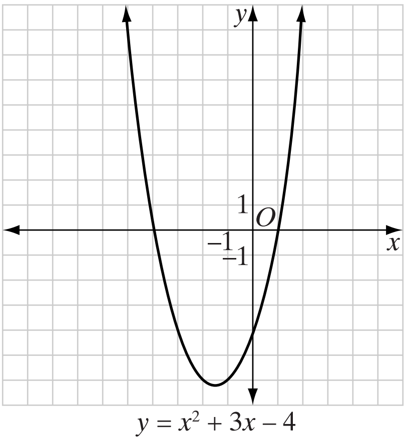
The x-coordinates of these points are the roots of $x^2 - 5x + 6 = 0$.

Answer The roots are 2 and 3. The solution set is $\{2, 3\}$.

Note that in Example 1 the quadratic expression $x^2 - 5x + 6$ can be factored into $(x - 2)(x - 3)$, from which we can obtain the solution set $\{2, 3\}$.

EXAMPLE 2

Use the graph of $y = x^2 + 3x - 4$ to find the linear factors of $x^2 + 3x - 4$.



Solution The graph intersects the x-axis at $(-4, 0)$ and $(1, 0)$.

Therefore, the roots are -4 and 1.

If $x = -4$, then $x - (-4) = (x + 4)$ is a factor.

If $x = 1$, then $(x - 1)$ is a factor.

Answer The linear factors of $x^2 + 3x - 4$ are $(x + 4)$ and $(x - 1)$.

EXERCISES

Writing About Mathematics

1. The coordinates of the vertex of a parabola $y = x^2 + 2x + 5$ are $(-1, 4)$. Does the equation $x^2 + 2x + 5 = 0$ have real roots? Explain your answer.

2. The coordinates of the vertex of a parabola $y = -x^2 - 2x + 3$ are $(-1, 4)$. Does the equation $-x^2 - 2x + 3 = 0$ have real roots? Explain your answer.

Developing Skills

In 3–10: **a.** Draw the graph of the parabola. **b.** Using the graph, find the real numbers that are elements of the solution set of the equation. **c.** Using the graph, factor the corresponding quadratic expression if possible.

3. $y = x^2 + 6x + 5; 0 = x^2 + 6x + 5$

4. $y = x^2 + 2x + 1; 0 = x^2 + 2x + 1$

5. $y = x^2 - 2x - 3; 0 = x^2 - 2x - 3$

6. $y = -x^2 + x + 2; 0 = -x^2 + x + 2$

7. $y = x^2 - 2x + 1; 0 = x^2 - 2x + 1$

8. $y = -x^2 + 3x - 2; 0 = -x^2 + 3x - 2$

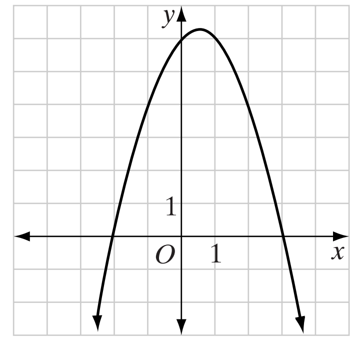
9. $y = x^2 + 4x + 5; 0 = x^2 + 4x + 5$

10. $y = -x^2 + 5x + 6; 0 = -x^2 + 5x + 6$

11. If the graph of a quadratic function, $f(x)$, crosses the x-axis at $x = 6$ and $x = 8$, what are two factors of $f(x)$?

12. If the factors of a quadratic function, $h(x)$, are $(x + 6)$ and $(x - 3)$, what is the solution set for the equation $h(x) = 0$?

In 13–15, refer to the graph of the parabola shown below.



13. Which of the following is the equation of the parabola?

(1) $y = (x + 2)(x - 3)$

(2) $y = -(x + 2)(x - 3)$

(3) $y = (x - 2)(x + 3)$

(4) $y = -(x - 2)(x + 3)$

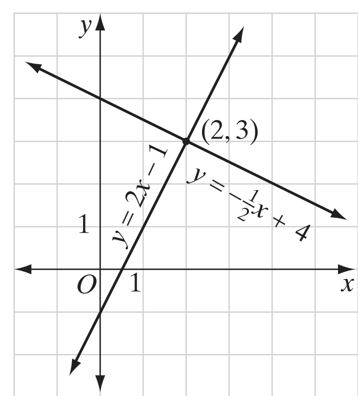
14. Set the equation of the parabola equal to 0. What are the roots of this quadratic equation?

15. If the graph of the parabola is reflected in the x-axis, how many roots will its corresponding quadratic equation have?

13-4 GRAPHIC SOLUTION OF A QUADRATIC-LINEAR SYSTEM

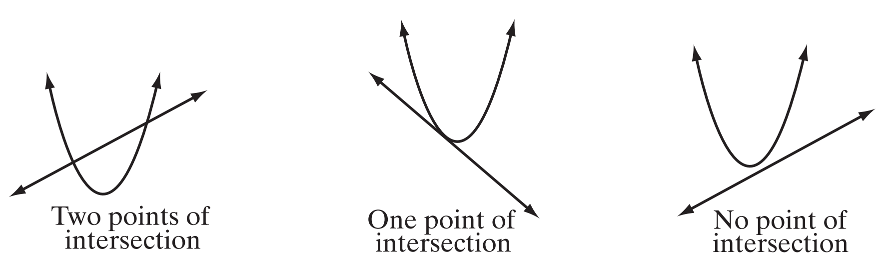
In Chapter 10 you learned how to solve a *system of linear equations* by graphing. For example, the graphic solution of the given system of linear equations is shown below.

$$y = -\tfrac{1}{2}x + 4$$
$$y = 2x - 1$$



Since the point of intersection, (2, 3), is a solution of both equations, the common solution of the system is $x = 2$ and $y = 3$.

A **quadratic-linear system** consists of a quadratic equation and a linear equation. The solution of a quadratic-linear system is the set of ordered pairs of numbers that make both equations true. As shown below, the line may intersect the curve in two, one, or no points. Thus the solution set may contain two ordered pairs, one ordered pair, or no ordered pairs.



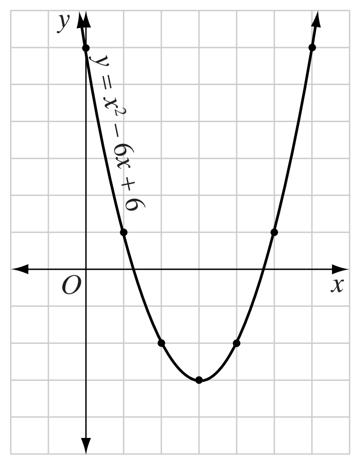
EXAMPLE 1

Solve the quadratic-linear system graphically: $y = x^2 - 6x + 6$
$y = x - 4$

Solution (1) Draw the graph of $y = x^2 - 6x + 6$.

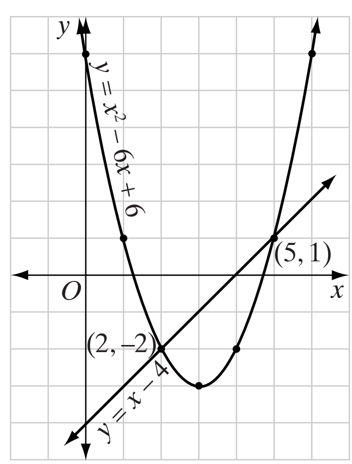
The axis of symmetry of a parabola is $x = \frac{-b}{2a}$. Therefore, the axis of symmetry for the graph of $y = x^2 - 6x + 6$ is $x = \frac{-(-6)}{2(1)}$ or $x = 3$. Make a table of values using integral values of x less than 3 and greater than 3. Plot the points associated with each pair (x, y) and join them with a smooth curve.

x	$x^2 - 6x + 6$	y
0	0 − 0 + 6	6
1	1 − 6 + 6	1
2	4 − 12 + 6	−2
3	9 − 18 + 6	−3
4	16 − 24 + 6	−2
5	25 − 30 + 6	1
6	36 − 36 + 6	6



(2) On the same set of axes, draw the graph of $y = x - 4$. Make a table of values and plot the points.

x	$x - 4$	y
0	0 − 4	−4
2	2 − 4	−2
4	4 − 4	0



The line could also have been graphed by using the slope $= 1$ or $\frac{1}{1}$, and the y-intercept, -4. Starting at the point $(0, -4)$ move 1 unit up and 1 unit to the right to locate a second point. Then again, move 1 unit up and 1 unit to the right to locate a third point. Draw a line through these points.

(3) Find the coordinates of the points at which the graphs intersect. The graphs intersect at $(2, -2)$ and at $(5, 1)$. Check each solution in each equation. Four checks are required in all. The checks are left for you.

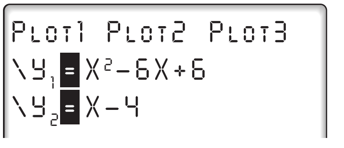
Calculator Solution (1) Enter the equations into the Y= menu.

ENTER: Y= X,T,Θ,n x² − 6

X,T,Θ,n + 6 ▼

X,T,Θ,n − 4

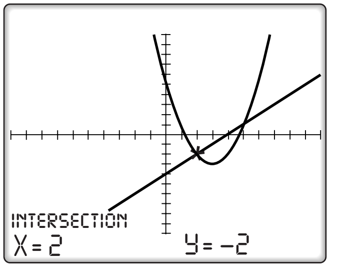
DISPLAY:



(2) Calculate the first intersection by choosing intersect from the CALC menu. Accept $Y^1 = X^2 - 6X + 6$ as the first curve and $Y^2 = X - 4$ as the second curve. Then press enter when the screen prompts "Guess?".

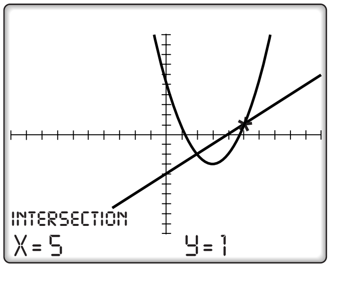
ENTER: 2nd CALC 5 ENTER ENTER ENTER

DISPLAY:



The calculator displays the coordinates of the intersection point at the bottom of the screen.

(3) To calculate the second intersection point, repeat the process from (2), but when the screen prompts "Guess?" move the cursor with the left and right arrow keys to the approximate position of the second intersection point.



Answer The solution set is $\{(2, -2), (5, 1)\}$.

EXERCISES

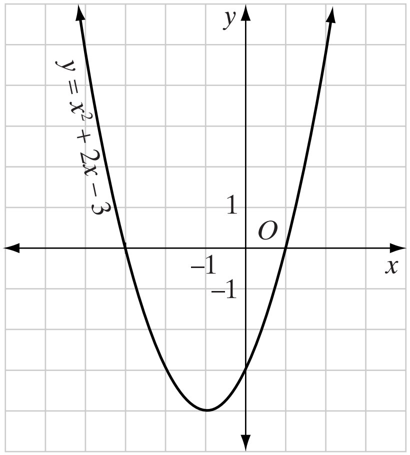
Writing About Mathematics

1. What is the solution set of a system of equations when the graphs of the equations do not intersect? Explain your answer.

2. Melody said that the equations $y = x^2$ and $y = -2$ do not have a common solution even before she drew their graphs. Explain how Melody was able to justify her conclusion.

Developing Skills

In 3–6, use the graph on the right to find the common solution of the system.



3. $y = x^2 + 2x - 3$
$y = 0$

4. $y = x^2 + 2x - 3$
$y = -3$

5. $y = x^2 + 2x - 3$
$y = -4$

6. $y = x^2 + 2x - 3$
$y = 5$

7. For what values of c do the equations $y = x^2 + 2x - 3$ and $y = c$ have no points in common?

8. a. Draw the graph of $y = x^2 - 4x - 2$, in the interval $-1 \leq x \leq 5$.

b. On the same set of axes, draw the graph of $y = -x - 2$.

c. Write the coordinates of the points of intersection of the graphs made in parts **a** and **b**.

d. Check the common solutions found in part **c** in both equations.

In 9–16, find graphically and check the solution set of each system of equations.

9. $y = x^2$
$y = x + 2$

10. $y = x^2 - 2x - 4$
$y = x$

11. $y = x^2 + 2x + 1$
$y = 2x + 5$

12. $y = 4x - x^2$
$y = x - 4$

13. $y = x^2 - 8x + 15$
$x + y = 5$

14. $y = -x^2 + 6x - 5$
$y = 3$

15. $y = x^2 + 4x + 1$
$y = 2x + 1$

16. $y = x^2 + x - 4$
$2x - y = 2$

Applying Skills

17. When a stone is thrown upward from a height of 5 feet with an initial velocity of 48 feet per second, the height of the stone y after x seconds is given by the function $y = -16x^2 + 48x + 5$.

a. Draw a graph of the given function. Let each horizontal unit equal $\frac{1}{2}$ second and each vertical unit equal 5 feet. Plot points for $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, and 3 seconds.

b. On the same set of axes, draw the graph of $y = 25$.

c. From the graphs drawn in parts **a** and **b**, determine when the stone is at a height of 25 feet.

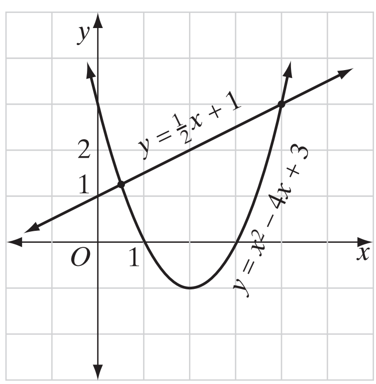
18. If you drop a baseball on Mars, the gravity accelerates the baseball at 12 feet per second squared. Let's suppose you drop a baseball from a height of 100 feet. A formula for the height, y, of the baseball after x seconds is given by $y = -6x^2 + 100$.

a. On a calculator, graph the given function. Set Xmin to –10, Xmax to 10, Xscl to 1, Ymin to −5, Ymax to 110, and Yscl to 10.

b. Graph the functions $y = 46$ and $y = 0$ as Y_2 and Y_3.

c. Using a calculator method, determine, to the *nearest tenth of a second*, when the baseball has a height of 46 feet and when the baseball hits the ground (reaches a height of 0 feet).

13-5 ALGEBRAIC SOLUTION OF A QUADRATIC-LINEAR SYSTEM



In the last section, we learned to solve a quadratic-linear system by finding the points of intersection of the graphs. The solutions of most of the systems that we solved were integers that were easy to read from the graphs. However, not all solutions are integers. For example, the graphs of $y = x^2 - 4x + 3$ and $y = \frac{1}{2}x + 1$ are shown at the right. They intersect in two points. One of those points, (4, 3), has integral coordinates and can be read easily from the graph. However, the coordinates of the other point are not integers, and we are not able to identify the exact values of x and y from the graph.

In Chapter 10 we learned that a system of linear equations can be solved by an algebraic method of substitution. This method can also be used for a quadratic-linear system. The algebraic solution of the system graphed on the previous page is shown in Example 1.

EXAMPLE 1

Solve algebraically and check: $y = x^2 - 4x + 3$
$y = \frac{1}{2}x + 1$

Solution

(1) Since y is expressed in terms of x in the linear equation, substitute the expression $\frac{1}{2}x + 1$ for y in the quadratic equation to form an equation in one variable:

$$y = x^2 - 4x + 3$$
$$\frac{1}{2}x + 1 = x^2 - 4x + 3$$

(2) To eliminate fractions as coefficients, multiply both sides of the equation by 2:

$$2\left(\frac{1}{2}x + 1\right) = 2(x^2 - 4x + 3)$$
$$x + 2 = 2x^2 - 8x + 6$$

(3) Write the quadratic equation in standard form:

$$0 = 2x^2 - 9x + 4$$

(4) Solve the quadratic equation by factoring:

$$0 = (2x - 1)(x - 4)$$

(5) Set each factor equal to 0 and solve for x.

$2x - 1 = 0$	$x - 4 = 0$
$2x = 1$	$x = 4$
$x = \frac{1}{2}$	

(6) Substitute each value of x in the linear equation to find the corresponding value of y:

$y = \frac{1}{2}x + 1$	$y = \frac{1}{2}x + 1$
$y = \frac{1}{2}\left(\frac{1}{2}\right) + 1$	$y = \frac{1}{2}(4) + 1$
$y = \frac{1}{4} + 1$	$y = 2 + 1$
$y = \frac{5}{4}$	$y = 3$

(7) Write each solution as coordinates: $(x, y) = \left(\frac{1}{2}, \frac{5}{4}\right)$ | $(x, y) = (4, 3)$

(8) Check each ordered pair in each of the given equations:

Check for $x = \frac{1}{2}, y = \frac{5}{4}$

$y = x^2 - 4x + 3$	$y = \frac{1}{2}x + 1$
$\frac{5}{4} \stackrel{?}{=} \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 3$	$\frac{5}{4} \stackrel{?}{=} \frac{1}{2}\left(\frac{1}{2}\right) + 1$
$\frac{5}{4} \stackrel{?}{=} \frac{1}{4} - 2 + 3$	$\frac{5}{4} \stackrel{?}{=} \frac{1}{4} + 1$
$\frac{5}{4} = \frac{5}{4}$ ✔	$\frac{5}{4} = \frac{5}{4}$ ✔

Check for $x = 4, y = 3$

$y = x^2 - 4x + 3$	$y = \frac{1}{2}x + 1$
$3 \stackrel{?}{=} (4)^2 - 4(4) + 3$	$3 \stackrel{?}{=} \frac{1}{2}(4) + 1$
$3 \stackrel{?}{=} 16 - 16 + 3$	$3 \stackrel{?}{=} 2 + 1$
$3 = 3$ ✔	$3 = 3$ ✔

Answer $\left\{\left(\frac{1}{2}, \frac{5}{4}\right), (4, 3)\right\}$

EXAMPLE 2

The length of the longer leg of a right triangle is 2 units more than twice the length of the shorter leg. The length of the hypotenuse is 13 units. Find the lengths of the legs of the triangle.

Solution Let a = the length of the longer leg and b = the length of the shorter leg.

(1) Use the Pythagorean Theorem to write an equation:

$$a^2 + b^2 = (13)^2$$
$$a^2 + b^2 = 169$$

(2) Use the information in the first sentence of the problem to write another equation:

$$a = 2b + 2$$

(3) Substitute the expression for a from step 2 in the equation in step 1:

$$a^2 + b^2 = 169$$
$$(2b + 2)^2 + b^2 = 169$$

(4) Square the binomial and write the equation in standard form:

$$4b^2 + 8b + 4 + b^2 = 169$$
$$5b^2 + 8b - 165 = 0$$

(5) Factor the left member of the equation:

$$(5b + 33)(b - 5) = 0$$

(6) Set each factor equal to 0 and solve for b:	$5b + 33 = 0$ $5b = -33$ $b = \frac{-33}{5}$	$b - 5 = 0$ $b = 5$ $b = 5$
(7) For each value of b, find the value of a. Use the linear equation in step 2:	$a = 2b + 2$ $a = 2\left(\frac{-33}{5}\right) + 2$	$a = 2b + 2$ $a = 2(5) + 2$
(8) Reject the negative values. Use the pair of positive values to write the answer.	$a = \frac{-56}{5}$	$a = 12$

Answer The lengths of the legs are 12 units and 5 units.

EXERCISES

Writing About Mathematics

1. Explain why the equations $y = x^2$ and $y = -4$ have no common solution in the set of real numbers.

2. Explain why the equations $x^2 + y^2 = 49$ and $x = 8$ have no common solution in the set of real numbers.

Developing Skills

In 3–20, solve each system of equations algebraically and check all solutions.

3. $y = x^2 - 2x$
   $y = x$

4. $y = x^2 + 5$
   $y = x + 5$

5. $y = x^2 - 4x + 3$
   $y = x - 1$

6. $y = x^2 + 2x + 1$
   $y = x + 3$

7. $x^2 + y = 9$
   $y = x + 9$

8. $x^2 + y = 2$
   $y = -x$

9. $x^2 + 2y = 5$
   $y = x + 1$

10. $y = 2x^2 - 6x + 5$
    $y = x + 2$

11. $y = 2x^2 + 2x + 3$
    $y - x = 3$

12. $y = 3x^2 - 8x + 5$
    $x + y = 3$

13. $y = x^2 + 3x + 1$
    $y = \frac{1}{3}x + 2$

14. $y = x^2 - 6x + 8$
    $y = -\frac{1}{2}x + 2$

15. $x^2 + y^2 = 25$
    $x = 2y - 5$

16. $x^2 + y^2 = 100$
    $y = x + 2$

17. $x^2 + y^2 = 50$
    $x = y$

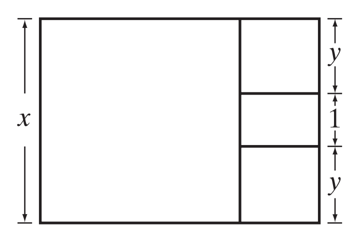
18. $x^2 + y^2 = 40$
    $y = 2x + 2$

19. $x^2 + y^2 = 20$
    $y = x + 2$

20. $x^2 + y^2 = 2$
    $y = x + 2$

Applying Skills

21. A rectangular tile pattern consists of a large square, two small squares, and a rectangle arranged as shown in the diagram. The height of the small rectangle is 1 unit and the area of the tile is 70 square units.



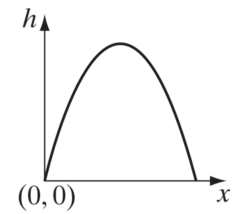
   a. Write a linear equation that expresses the relationship between the length of the sides of the quadrilaterals that make up the tile.

   b. Write a second-degree equation using the sum of the areas of the quadrilaterals that make up the tile.

   c. Solve algebraically the system of equations written in parts **a** and **b**.

   d. What are the dimensions of each quadrilateral in the tile?

22. A doorway to a store is in the shape of an arch whose equation is $h = -\frac{3}{4}x^2 + 6x$, where x represents the horizontal distance, in feet, from the left end of the base of the doorway and h is the height, in feet, of the doorway x feet from the left end of the base.



a. How wide is the doorway at its base?

b. What is the maximum height of the doorway?

c. Can a box that is 6 feet wide, 6 feet long, and 5 feet high be moved through the doorway? Explain your answer.

23. The length of the diagonal of a rectangle is $\sqrt{85}$ meters. The length of the rectangle is 1 meter longer than the width. Find the dimensions of the rectangle.

24. The length of one leg of an isosceles triangle is $\sqrt{29}$ feet. The length of the altitude to the base of the triangle is 1 foot more than the length of the base

a. Let a = the length of the altitude to the base and b = the distance from the vertex of a base angle to the vertex of the right angle that the altitude makes with the base. Use the Pythagorean Theorem to write an equation in terms of a and b.

b. Represent the length of the base in terms of b.

c. Represent the length of the altitude, a, in terms of b.

d. Solve the system of equations from parts **a** and **c**.

e. Find the length of the base and the length of the altitude to the base.

f. Find the perimeter of the triangle.

g. Find the area of the triangle.

CHAPTER SUMMARY

The equation $y = ax^2 + bx + c$, where $a \neq 0$, is a **quadratic function** whose domain is the set of real numbers and whose graph is a **parabola**. The **axis of symmetry of the parabola** is the vertical line $x = \frac{-b}{2a}$. The **vertex** or **turning point** of the parabola is on the axis of symmetry. If $a > 0$, the parabola opens upward and the y-value of the vertex is a **minimum** value for the range of the function. If $a < 0$, the parabola opens downward and the y-value of the vertex is a **maximum** value for the range of the function.

A **quadratic-linear system** consists of two equations one of which is an equation of degree two and the other a linear equation (an equation of degree one). The common solution of the system may be found by graphing the equations on the same set of axes or by using the algebraic method of substitution. A quadratic-linear system of two equations may have two, one, or no common solutions.

The **roots of the equation** $ax^2 + bx + c = 0$ are the x-coordinates of the points at which the function $y = ax^2 + bx + c$ intersects the x-axis. The real number k is a root of $ax^2 + bx + c = 0$ if and only if $(x - k)$ is a factor of $ax^2 + bx + c$.

When the graph of $y = x^2$ is translated by k units in the vertical direction, the equation of the image is $y = x^2 + k$. When the graph of $y = x^2$ is translated k units in the horizontal direction, the equation of the image is $y = (x - k)^2$. When the graph of $y = x^2$ is reflected over the x-axis, the equation of the image is $y = -x^2$. The graph of $y = kx^2$ is the result of stretching the graph of $y = x^2$ in the vertical direction when $k > 1$ or of compressing the graph of $y = x^2$ when $0 < k < 1$.

VOCABULARY

13-1 Standard form • Polynomial equation of degree two • Quadratic equation • Roots of an equation • Double root

13-2 Parabola • Second-degree polynomial function • Quadratic function • Minimum • Turning point • Vertex • Axis of symmetry of a parabola • Maximum

13-4 Quadratic-linear system

REVIEW EXERCISES

1. Explain why $x = y^2$ is not a function when the domain and range are the set of real numbers.
2. Explain why $x = y^2$ is a function when the domain and range are the set of positive real numbers.

In 3–6, the set of ordered pairs of a relation is given. For each relation, **a.** list the elements of the domain, **b.** list the elements of the range, **c.** determine if the relation is a function.

3. $\{(1, 3), (2, 2), (3, 1), (4, 0), (5, -1)\}$
4. $\{(1, 3), (1, 2), (1, 1), (1, 0), (1, -1)\}$
5. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$
6. $\{(0, 1), (1, 1), (2, 1), (3, 1), (4, 1)\}$

In 7–10, for each of the given functions:

a. Write the equation of the axis of symmetry.

b. Draw the graph.

c. Write the coordinates of the turning point.

d. Does the function have a maximum or a minimum?

e. What is the range of the function?

7. $y = x^2 - 6x + 6$

8. $y = x^2 - 4x - 1$

9. $f(x) = -x^2 - 2x + 6$

10. $f(x) = -x^2 + 6x - 1$

In 11–16, solve each system of equations graphically and check the solution(s) if they exist.

11. $y = x^2 - 6$
$x + y = 6$

12. $y = -x^2 + 2x + 1$
$y = x - 5$

13. $y = x^2 - x - 3$
$y = x$

14. $y = x^2 - 4x$
$x + y = 4$

15. $y = 5 - x^2$
$y = 4$

16. $y = 2x - x^2$
$x + y = 2$

In 17–22, solve each system of equations algebraically and check the solutions.

17. $x^2 - y = 5$
$y = 3x - 1$

18. $y = x^2 - 4x + 9$
$y - 1 = 2x$

19. $x^2 - 2y = 11$
$y = x - 4$

20. $y = x^2 - 6x + 5$
$y = x - 1$

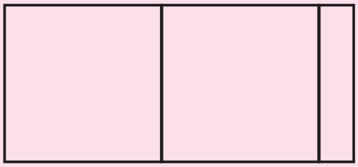
21. $x^2 + y^2 = 40$
$y = 3x$

22. $x^2 + y^2 = 5$
$y = \frac{1}{2}x$

23. Write an equation for the resulting function if the graph of $y = x^2$ is shifted 3 units up, 2.5 units to the left, and is reflected over the x-axis.

24. The sum of the areas of two squares is 85. The length of a side of the larger square minus the length of a side of the smaller square is 1. Find the length of a side of each square.

25. Two square pieces are cut from a rectangular piece of carpet as shown in the diagram. The area of the original piece is 144 square feet, and the width of the small rectangle that is left is 2 feet. Find the dimensions of the original piece of carpet.



Exploration

Write the square of each integer from 2 to 20. Write the prime factorization of each of these squares. What do you observe about the prime factorization of each of these squares?

Let n be a positive integer and a, b, and c be prime numbers. If the prime factorization of $n = a^3 \times b^2 \times c^4$, is n a perfect square? Is $\sqrt{n}$ rational or irrational? Express $\sqrt{n}$ in terms of a, b, and c.

CUMULATIVE REVIEW — CHAPTERS 1–13

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. When $a = -7$, $5 - 2a^2$ equals
(1) 147 (2) -93 (3) 103 (4) -147

2. Which expression is rational?
(1) π (2) $\sqrt{\frac{1}{3}}$ (3) $\sqrt{\frac{2}{8}}$ (4) $\sqrt{0.4}$

3. When factored completely, $3x^2 - 75$ can be expressed as
(1) $(3x + 15)(x - 5)$ (3) $3x(x + 5)(x - 5)$
(2) $(x + 5)(3x - 15)$ (4) $3(x + 5)(x - 5)$

4. When $b^2 + 4b$ is subtracted from $3b^2 - 3b$ the difference is
(1) $3 - 7b$ (2) $-2b^2 + 7b$ (3) $2b^2 - 7b$ (4) $2b^2 + b$

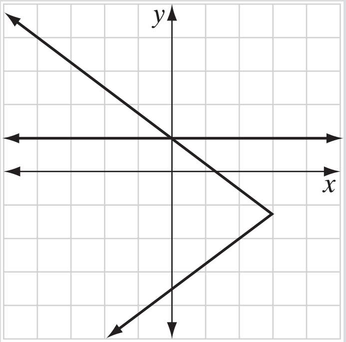
5. The solution set of the equation $0.5x + 4 = 2x - 0.5$ is
(1) {3} (2) {1.4} (3) {30} (4) {14}

6. Which of these represents the quadratic function $y = x^2 + 5$ shifted 2 units down and 4 units to the right?
(1) $y = (x - 4)^2 - 2$ (3) $y = (x + 4)^2 + 7$
(2) $y = (x - 4)^2 + 3$ (4) $y = (x - 2)^2 + 9$

7. The slope of the line whose equation is $x - 2y = 4$ is
(1) -2 (2) 2 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

8. The solution set of the equation $x^2 - 7x + 10 = 0$ is
(1) {–2, –5} (3) $(x - 2)(x - 5)$
(2) {2, 5} (4) $(x + 2)(x + 5)$

9. Which of these shows the graph of a linear function intersecting the graph of an absolute value function?

(1)


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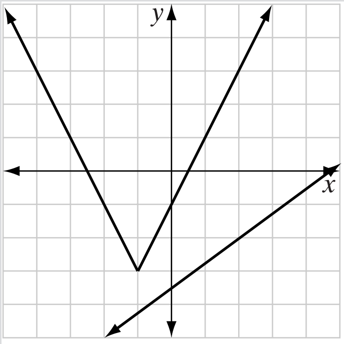

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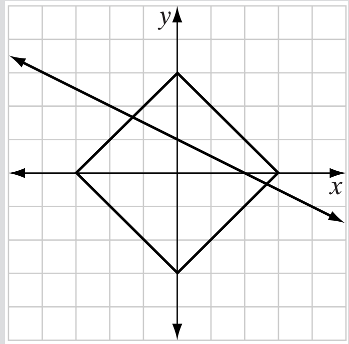

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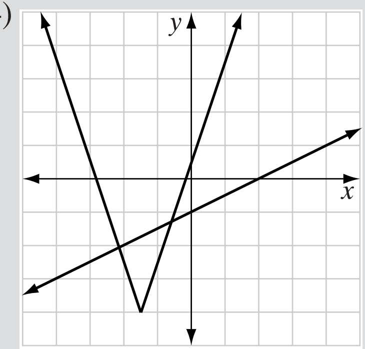

10. In $\triangle ABC$, $m\angle A = 72$ and $m\angle B = 83$. What is the measure of $\angle C$?
(1) 155° (2) 108° (3) 97° (4) 25°

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. In a class of 300 students, 242 take math, 208 take science, and 183 take both math and science. How many students take neither math nor science?

12. Each of the equal sides of an isosceles triangle is 3 centimeters longer than the base. The perimeter of the triangle is 54 centimeters, what is the measure of each side of the triangle?

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. The base of a right circular cylinder has a diameter of 5.00 inches. Sally measured the circumference of the base of the cylinder and recorded it to be 15.5 inches. What is the percent of error in her measurement? Express your answer to the nearest tenth of a percent.

14. Solve the following system of equations and check your solutions.

$$y = -x^2 + 3x + 1$$
$$y = x + 1$$

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Find to the nearest degree the measure of the acute angle that the graph of $y = 2x - 4$ makes with the x-axis.

16. Jean Forester has a small business making pies and cakes. Today, she must make at least 4 cakes to fill her orders and at least 3 pies. She has time to make a total of no more than 10 pies and cakes.

a. Let x represent the number of cakes that Jean makes and y represent the number of pies. Write three inequalities that can be used to represent the number of pies and cakes that she can make.

b. In the coordinate plane, graph the inequalities that you wrote and indicate the region that represents their common solution.

c. Write at least three ordered pairs that represent the number of pies and cakes that Jean can make.