CHAPTER

6

RATIO AND PROPORTION

Everyone likes to save money by purchasing something at a reduced price. Because merchants realize that a reduced price may entice a prospective buyer to buy on impulse or to buy at one store rather than another, they offer discounts and other price reductions. These discounts are often expressed as a percent off of the regular price.

When the Acme Grocery offers a 25% discount on frozen vegetables and the Shop Rite Grocery advertises "Buy four, get one free," the price-conscious shopper must decide which is the better offer if she intends to buy five packages of frozen vegetables.

In this chapter, you will learn how ratios, and percents which are a special type of ratio, are used in many everyday problems.

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6-1 RATIO

A **ratio**, which is a comparison of two numbers by division, is the quotient obtained when the first number is divided by the second, nonzero number.

Since a ratio is the quotient of two numbers divided in a definite order, care must be taken to write each ratio in its intended order. For example, the ratio of 3 to 1 is written

$\frac{3}{1}$ (as a fraction) or 3 : 1 (using a colon)

while the ratio of 1 to 3 is written

$\frac{1}{3}$ (as a fraction) or 1 : 3 (using a colon)

In general, the ratio of a to b can be expressed as

$$\frac{a}{b} \qquad \text{or} \qquad a \div b \qquad \text{or} \qquad a : b$$

To find the ratio of two quantities, both quantities must be expressed in the same unit of measure before their quotient is determined. For example, to compare the value of a nickel and a penny, we first convert the nickel to 5 pennies and then find the ratio, which is $\frac{5}{1}$ or 5 : 1. Therefore, a nickel is worth 5 times as much as a penny. The ratio has no unit of measure.

Equivalent Ratios

Since the ratio $\frac{5}{1}$ is a fraction, we can use the multiplication property of 1 to find many **equivalent ratios**. For example:

$$\frac{5}{1} = \frac{5}{1} \times \frac{2}{2} = \frac{10}{2} \qquad \frac{5}{1} = \frac{5}{1} \times \frac{3}{3} = \frac{15}{3} \qquad \frac{5}{1} = \frac{5}{1} \times \frac{x}{x} = \frac{5x}{1x}$$

$$(x \neq 0)$$

From the last example, we see that $5x$ and $1x$ represent two numbers whose ratio is 5 : 1.

In general, if a, b, and x are numbers ($b \neq 0, x \neq 0$), ax and bx represent two numbers whose ratio is $a : b$ because

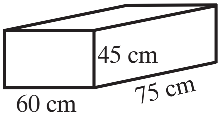
$$\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times \frac{x}{x} = \frac{ax}{bx}$$

Also, since a ratio such as $\frac{24}{16}$ is a fraction, we can divide the numerator and the denominator of the fraction by the same nonzero number to find equivalent ratios. For example:

$$\frac{24}{16} = \frac{24 \div 2}{16 \div 2} = \frac{12}{8} \qquad \frac{24}{16} = \frac{24 \div 4}{16 \div 4} = \frac{6}{4} \qquad \frac{24}{16} = \frac{24 \div 8}{16 \div 8} = \frac{3}{2}$$

A ratio is expressed in **simplest form** when both terms of the ratio are whole numbers and when there is no whole number other than 1 that is a factor of both of these terms. Therefore, to express the ratio $\frac{24}{16}$ in simplest form, we divide both terms by 8, the largest integer that will divide both 24 and 16. Therefore, $\frac{24}{16}$ in simplest form is $\frac{3}{2}$.

Continued Ratio



Comparisons can also be made for three or more quantities. For example, the length of a rectangular solid is 75 centimeters, the width is 60 centimeters, and the height is 45 centimeters. The ratio of the length to the width is 75 : 60, and the ratio of the width to the height is 60 : 45. We can write these two ratios in an abbreviated form as the continued ratio 75 : 60 : 45.

A **continued ratio** is a comparison of three or more quantities in a definite order. Here, the ratio of the measures of the length, width, and height (in that order) of the rectangular solid is 75 : 60 : 45 or, in simplest form, 5 : 4 : 3.

► **In general, the ratio of the numbers a, b, and c ($b \neq 0$, $c \neq 0$) is $a : b : c$.**

EXAMPLE 1

An oil tank with a capacity of 200 gallons contains 50 gallons of oil.

a. Find the ratio of the number of gallons of oil in the tank to the capacity of the tank.

b. What part of the tank is full?

Solution

a. Ratio $= \frac{\text{number of gallons of oil in tank}}{\text{capacity of tank}} = \frac{50}{200} = \frac{1}{4}$.

b. The tank is $\frac{1}{4}$ full.

Answers **a.** $\frac{1}{4}$ **b.** $\frac{1}{4}$ full

EXAMPLE 2

Compute the ratio of 6.4 ounces to 1 pound.

Solution First, express both quantities in the same unit of measure. Use the fact that 1 pound = 16 ounces.

$$\frac{6.4\text{ ounces}}{1\text{ pound}} = \frac{6.4\text{ ounces}}{16\text{ ounces}} = \frac{6.4}{16} = \frac{6.4}{16} \times \frac{10}{10} = \frac{64}{160} = \frac{64 \div 32}{160 \div 32} = \frac{2}{5}$$

Calculator Solution On a calculator, divide 6.4 ounces by 16 ounces.

ENTER: 6.4 **÷** 16 **ENTER**

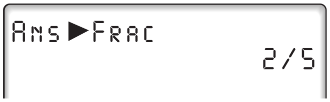
DISPLAY:

```
6.4/16
                .4
```

Change the decimal in the display to a fraction.

ENTER: **2nd** **ANS** **MATH** **ENTER** **ENTER**

DISPLAY:



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Ans►Frac
               2/5
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Answer The ratio is 2 : 5.

EXAMPLE 3

Express the ratio $1\frac{3}{4}$ to $1\frac{1}{2}$ in simplest form.

Solution Since a ratio is the quotient obtained when the first number is divided by the second, divide $1\frac{3}{4}$ by $1\frac{1}{2}$.

$$1\frac{3}{4} \div 1\frac{1}{2} = \frac{7}{4} \div \frac{3}{2} = \frac{7}{4} \cdot \frac{2}{3} = \frac{14}{12} = \frac{7}{6}$$

Answer The ratio in simplest form is $\frac{7}{6}$ or 7 : 6.

EXERCISES

Writing About Mathematics

1. Last week, Melanie answered 24 out of 30 questions correctly on a test. This week she answered 20 out of 24 questions correctly. On which test did Melanie have better results? Explain your answer.

2. Explain why the ratio 1.5 : 4.5 is not in simplest form.

Developing Skills

In 3–12, express each ratio in simplest form: **a.** as a fraction **b.** using a colon

3. 36 to 12 **4.** 48 to 24 **5.** 40 to 25 **6.** 12 to 3 **7.** 5 to 4

8. 8 to 32 **9.** 40 to 5 **10.** 0.2 to 8 **11.** 72 to 1.2 **12.** $3c$ to $5c$

13. If the ratio of two numbers is 10 : 1, the larger number is how many times the smaller number?

14. If the ratio of two numbers is 8 : 1, the smaller number is what fractional part of the larger number?

In 15–19, express each ratio in simplest form.

15. $\frac{3}{4}$ to $\frac{1}{4}$

16. $1\frac{1}{8}$ to $\frac{3}{8}$

17. 1.2 to 2.4

18. 0.75 to 0.25

19. 6 to 0.25

In 20–31, express each ratio in simplest form.

20. 80 m to 16 m

21. 75 g to 100 g

22. 36 cm to 72 cm

23. 54 g to 90 g

24. 75 cm to 350 cm

25. 8 ounces to 1 pound

26. $1\frac{1}{2}$ hr to $\frac{1}{2}$ hr

27. 3 in. to $\frac{1}{2}$ in.

28. 1 ft to 1 in.

29. 1 yd to 1 ft

30. 1 hr to 15 min

31. 6 dollars to 50 cents

Applying Skills

32. A baseball team played 162 games and won 90.

a. What is the ratio of the number of games won to the number of games played?

b. For every nine games played, how many games were won?

33. A student did six of ten problems correctly.

a. What is the ratio of the number right to the number wrong?

b. For every two answers that were wrong, how many answers were right?

34. A cake recipe calls for $1\frac{1}{4}$ cups of milk to $1\frac{3}{4}$ cups of flour. Write, in simplest form, the ratio of the number of cups of milk to the number of cups of flour in this recipe.

35. The perimeter of a rectangular garden is 30 feet, and the width is 5 feet. Find the ratio of the length of the rectangle to its width in simplest form.

36. In a freshman class, there are b boys and g girls. Express the ratio of the number of boys to the total number of pupils.

37. The length of a rectangular classroom is represented by $3x$ and its width by $2x$. Find the ratio of the width of the classroom to its perimeter.

38. The ages of three teachers are 48, 28, and 24 years. Find, in simplest form, the continued ratio of these ages from oldest to youngest.

39. A woodworker is fashioning a base for a trophy. He starts with a block of wood whose length is twice its width and whose height is one-half its width. Write, in simplest form, the continued ratio of length to width to height.

40. Taya and Jed collect coins. The ratio of the number of coins in their collections, in some order, is 4 to 3. If Taya has 60 coins in her collection, how many coins could Jed have?

6-2 USING A RATIO TO EXPRESS A RATE

When two quantities have the same unit of measure, their ratio has no unit of measure. A **rate**, like a ratio, is a comparison of two quantities, but the quantities may have *different units of measures* and their ratio has a unit of measure.

For example, if a plane flies 1,920 kilometers in 3 hours, its *rate* of speed is a ratio that compares the distance traveled to the time that the plane was in flight.

$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{1{,}920\text{ kilometers}}{3\text{ hours}} = \frac{640\text{ kilometers}}{1\text{ hour}} = 640\text{ km/h}$$

The abbreviation km/h is read "kilometers per hour."

A rate may be expressed in **lowest terms** when the numbers in its ratio are whole numbers with no common factor other than 1. However, a rate is most frequently written as a ratio with 1 as its second term. As shown in the example above, the second term may be omitted when it is 1. A rate that has a denominator of 1 is called a **unit rate**. A rate that identifies the cost of an item per unit is called the **unit price**. For example, $0.15 per ounce or $3.79 per pound are unit prices.

EXAMPLE 1

Kareem scored 175 points in seven basketball games. Express, in lowest terms, the average rate of the number of points Kareem scored per game.

Solution

$$\text{Rate} = \frac{175\text{ points}}{7\text{ games}} = \frac{25\text{ points}}{1\text{ game}} = 25\text{ points per game}$$

Answer Kareem scored points at an average rate of 25 points per game. ◻

EXAMPLE 2

There are 5 grams of salt in 100 cubic centimeters of a solution of salt and water. Express, in lowest terms, the ratio of the number of grams of salt per cubic centimeters in the solution.

Solution

$$\frac{5\text{ g}}{100\text{ cm}^3} = \frac{1\text{ g}}{20\text{ cm}^3} = \frac{1}{20}\text{ g/cm}^3$$

Answer The solution contains $\frac{1}{20}$ grams or 0.05 grams of salt per cubic centimeter of solution. ◻

EXERCISES

Writing About Mathematics

1. How is a rate a special kind of ratio?

2. How does the way in which a rate is usually expressed differ from a ratio in simplest form?

Developing Skills

In 3–8, express each rate in lowest terms.

3. The ratio of 36 apples to 18 people.

4. The ratio of 48 patients to 6 nurses.

5. The ratio of \$1.50 to 3 liters.

6. The ratio of 96 cents to 16 grams.

7. The ratio of \$2.25 to 6.75 ounces.

8. The ratio of 62 miles to 100 kilometers.

Applying Skills

In 9–12, in each case, find the average rate of speed, expressed in miles per hour.

9. A vacationer traveled 230 miles in 4 hours.

10. A post office truck delivered mail on a 9-mile route in 2 hours.

11. A commuter drove 48 miles to work in $1\frac{1}{2}$ hours.

12. A race-car driver traveled 31 miles in 15 minutes. (Use 15 minutes $= \frac{1}{4}$ hour.)

13. If there are 240 tennis balls in 80 cans, how many tennis balls are in each can?

14. If an 11-ounce can of shaving cream costs 88 cents, what is the unit cost of the shaving cream in the can?

15. In a supermarket, the regular size of CleanRight cleanser contains 14 ounces and costs 49 cents. The giant size of CleanRight cleanser, which contains 20 ounces, costs 66 cents.

a. Find, correct to the *nearest tenth* of a cent, the cost per ounce for the regular can.

b. Find, correct to the *nearest tenth* of a cent, the cost per ounce for the giant can.

c. Which is the better buy?

16. Johanna and Al use computers for word processing. Johanna can keyboard 920 words in 20 minutes, and Al can keyboard 1,290 words in 30 minutes. Who is faster at entering words on a keyboard?

17. Ronald runs 300 meters in 40 seconds. Carlos runs 200 meters in 30 seconds. Who is the faster runner for short races?

6-3 VERBAL PROBLEMS INVOLVING RATIO

Any pair of numbers in the ratio 3 : 5 can be found by multiplying 3 and 5 by the same nonzero number.

$$\begin{array}{llll} 3(3) = 9 & 3(7) = 21 & 3(0.3) = 0.9 & 3(x) = 3x \\ 5(3) = 15 & 5(7) = 35 & 5(0.3) = 1.5 & 5(x) = 5x \\ 3:5 = 9:15 & 3:5 = 21:35 & 3:5 = 0.9:1.5 & 3:5 = 3x:5x \end{array}$$

Thus, for any nonzero number x, $3x : 5x = 3 : 5$.

In general, when we know the ratio of two or more numbers, we can use the terms of the ratio and a nonzero variable, x, to express the numbers. Any two numbers in the ratio $a : b$ can be written as ax and bx where x is a nonzero real number.

EXAMPLE 1

The perimeter of a triangle is 60 feet. If the sides are in the ratio 3 : 4 : 5, find the length of each side of the triangle.

Solution Let $3x$ = the length of the first side,

$4x$ = the length of the second side,

$5x$ = the length of the third side.

The perimeter of the triangle is 60 feet.

$$3x + 4x + 5x = 60$$
$$12x = 60$$
$$x = 5$$
$$3x = 3(5) = 15$$
$$4x = 4(5) = 20$$
$$5x = 5(5) = 25$$

Check

$15 : 20 : 25 = 3 : 4 : 5$ ✔

$15 + 20 + 25 = 60$ ✔

Answer The lengths of the sides are 15 feet, 20 feet, and 25 feet.

EXAMPLE 2

Two numbers have the ratio 2 : 3. The larger is 30 more than $\frac{1}{2}$ of the smaller. Find the numbers.

Solution Let $2x$ = the smaller number,

$3x$ = the larger number.

The larger number is 30 more than $\frac{1}{2}$ of the smaller number.

$$3x = \tfrac{1}{2}(2x) + 30$$
$$3x = x + 30$$
$$2x = 30$$
$$x = 15$$
$$2x = 2(15) = 30$$
$$3x = 3(15) = 45$$

Check
The ratio 30 : 45 in lowest terms is 2 : 3. ✔

One-half of the smaller number, 30, is 15. The larger number, 45, is 30 more than 15. ✔

Answer The numbers are 30 and 45.

EXERCISES

Writing About Mathematics

1. Two numbers in the ratio 2 : 3 can be written as $2x$ and $3x$. Explain why x cannot equal zero.

2. The ratio of the length of a rectangle to its width is 7 : 4. Pete said that the ratio of the length to the perimeter is 7 : 11. Do you agree with Pete? Explain why or why not.

Developing Skills

3. Two numbers are in the ratio 4 : 3. Their sum is 70. Find the numbers.

4. Find two numbers whose sum is 160 and that have the ratio 5 : 3.

5. Two numbers have the ratio 7 : 5. Their difference is 12. Find the numbers.

6. Find two numbers whose ratio is 4 : 1 and whose difference is 36.

7. The lengths of the sides of a triangle are in the ratio of 6 : 6 : 5. The perimeter of the triangle is 34 centimeters. Find the length of each side of the triangle.

8. The perimeter of a triangle is 48 centimeters. The lengths of the sides are in the ratio 3 : 4 : 5. Find the length of each side.

9. The perimeter of a rectangle is 360 centimeters. If the ratio of its length to its width is 11 : 4, find the dimensions of the rectangle.

10. The sum of the measures of two angles is 90°. The ratio of the measures of the angles is 2 : 3. Find the measure of each angle.

11. The sum of the measures of two angles is 180°. The ratio of the measures of the angles is 4 : 5. Find the measure of each angle.

12. The ratio of the measures of the three angles of a triangle is 2 : 2 : 5. Find the measures of each angle.

13. In a triangle, two sides have the same length. The ratio of each of these sides to the third side is 5 : 3. If the perimeter of the triangle is 65 inches, find the length of each side of the triangle.

14. Two positive numbers are in the ratio 3 : 7. The larger exceeds the smaller by 12. Find the numbers.

15. Two numbers are in the ratio 3 : 5. If 9 is added to their sum, the result is 41. Find the numbers.

Applying Skills

16. A piece of wire 32 centimeters in length is divided into two parts that are in the ratio 3 : 5. Find the length of each part.

17. The ratio of the number of boys in a school to the number of girls is 11 to 10. If there are 525 pupils in the school, how many of them are boys?

18. The ratio of Carl's money to Donald's money is 7 : 3. If Carl gives Donald $20, the two then have equal amounts. Find the original amount that each one had.

19. In a basketball free-throw shooting contest, the points made by Sam and Wilbur were in the ratio 7 : 9. Wilbur made 6 more points than Sam. Find the number of points made by each.

20. A chemist wishes to make $12\frac{1}{2}$ liters of an acid solution by using water and acid in the ratio 3 : 2. How many liters of each should she use?

6-4 PROPORTION

A **proportion** is an equation that states that two ratios are equal. Since the ratio 4 : 20 or $\frac{4}{20}$ is equal to the ratio 1 : 5 or $\frac{1}{5}$, we may write the proportion

$$4 : 20 = 1 : 5 \qquad \text{or} \qquad \frac{4}{20} = \frac{1}{5}$$

Each of these proportions is read as "4 is to 20 as 1 is to 5." The general form of a proportion may be written as:

$$\boldsymbol{a : b = c : d} \qquad \text{or} \qquad \boldsymbol{\frac{a}{b} = \frac{c}{d}\ (b \neq 0, d \neq 0)}$$

Each of these proportions is read as "*a* is to *b* as *c* is to *d*." There are four terms in this proportion, namely, *a*, *b*, *c*, and *d*. The outer terms, *a* and *d*, are called the **extremes** of the proportion. The inner terms, *b* and *c*, are the **means**.

$$\underbrace{a : \overbrace{b = c}^{\text{means}} : d}_{\text{extremes}} \qquad \text{or} \qquad \frac{\overset{\substack{\text{extreme}\\ \downarrow}}{a}}{\underset{\substack{\uparrow\\ \text{mean}}}{b}} = \frac{\overset{\substack{\text{mean}\\ \downarrow}}{c}}{\underset{\substack{\uparrow\\ \text{extreme}}}{d}}$$

In the proportion, $4 : 20 = 1 : 5$, the product of the means, $20(1)$, is equal to the product of the extremes, $4(5)$.

In the proportion, $\frac{5}{15} = \frac{10}{30}$, the product of the means, $15(10)$, is equal to the product of the extremes, $5(30)$.

In any proportion $\frac{a}{b} = \frac{c}{d}$, we can show that the product of the means is equal to the product of the extremes, $ad = bc$. Since $\frac{a}{b} = \frac{c}{d}$ is an equation, we can multiply both members by bd, the least common denominator of the fractions in the equation.

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ bd\left(\frac{a}{b}\right) &= bd\left(\frac{c}{d}\right) \\ \cancel{b}d\left(\frac{a}{\cancel{b}}\right) &= b\cancel{d}\left(\frac{c}{\cancel{d}}\right) \\ d \cdot a &= b \cdot c \\ ad &= bc \end{aligned}$$

Therefore, we have shown that the following statement is always true:

► **In a proportion, the product of the means is equal to the product of the extremes.**

Notice that the end result, $ad = bc$, is the result of multiplying the terms that are cross-wise from each other:

This is called **cross-multiplying**, which we have just shown to be valid.

If the product of two cross-wise terms is called a **cross product**, then the following is also true:

► **In a proportion, the cross products are equal.**

If a, b, c, and d are nonzero numbers and $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. There are three other proportions using a, b, c and d for which $ad = bc$.

$$\frac{a}{c} = \frac{b}{d} \qquad \frac{d}{b} = \frac{c}{a} \qquad \frac{d}{c} = \frac{b}{a}$$

For example, we know that $\frac{6}{4} = \frac{15}{10}$ is a proportion because $6(10) = 4(15)$. Therefore, each of the following is also a proportion.

$$\frac{6}{15} = \frac{4}{10} \qquad \frac{10}{4} = \frac{15}{6} \qquad \frac{10}{15} = \frac{4}{6}$$

EXAMPLE 1

Show that $\frac{4}{16} = \frac{5}{20}$ is a proportion.

Solution Three methods are shown here. The first two use paper and pencil; the last makes use of a calculator.

METHOD 1 Reduce each ratio to simplest form.

$$\frac{4}{16} = \frac{4 \div 4}{16 \div 4} = \frac{1}{4} \quad \text{and} \quad \frac{5}{20} = \frac{5 \div 5}{20 \div 5} = \frac{1}{4}$$

Since each ratio equals $\frac{1}{4}$, the ratios are equal and $\frac{4}{16} = \frac{5}{20}$ is a proportion.

METHOD 2 Show that the cross products are equal.

$$\frac{4}{16} = \frac{5}{20}$$

$$16 \times 5 = 4 \times 20$$

$$80 = 80$$

Therefore, $\frac{4}{16} = \frac{5}{20}$ is a proportion.

METHOD 3 Use a calculator. Enter the proportion. If the ratios are equal, then the calculator will display 1. If the ratios are not equal, the calculator will display 0.

ENTER: 4 **÷** 16 **2nd** **TEST** **ENTER** 5 **÷** 20 **ENTER**

DISPLAY:

```
4/16=5/20
                1
```

Since the calculator displays 1, the statement is true. The ratios are equal and $\frac{4}{16} = \frac{5}{20}$ is a proportion.

Answer Any one of the three methods shows that $\frac{4}{16} = \frac{5}{20}$ is a proportion. ▢

EXAMPLE 2

Solve the proportion $25 : q = 5 : 2$ for q.

Solution Since $25 : q = 5 : 2$ is a proportion, the product of the means is equal to the product of the extremes. Therefore:

$$\underbrace{25 : \overbrace{q = 5}^{\text{means}} : 2}_{\text{extremes}}$$

$$\underbrace{5q}_{\text{means}} = \underbrace{25(2)}_{\text{extremes}}$$

$$5q = 50$$

$$q = 10$$

Check

Reduce each ratio to simplest form.

$$25 : q = 5 : 2$$

$$25 : \mathbf{10} \overset{?}{=} 5 : 2$$

$$5 : 2 = 5 : 2 \checkmark$$

Answer $q = 10$

Note: Example 2 could also have been solved by setting up the proportion $\frac{25}{q} = \frac{5}{2}$ and then using cross-multiplication to solve for the variable.

EXAMPLE 3

Solve for x: $\frac{12}{x - 2} = \frac{32}{x + 8}$

Solution Use the fact that the product of the means equals the product of the extremes (the cross products are equal).

$$\frac{12}{x - 2} = \frac{32}{x + 8}$$

$$32(x - 2) = 12(x + 8)$$

$$\begin{array}{rcr} 32x - 64 & = & 12x + 96 \\ -12x + 64 & & -12x + 64 \\ \hline 20x & = & 160 \end{array}$$

$$\frac{20x}{20} = \frac{160}{20}$$

$$x = 8$$

Check

$$\frac{12}{x - 2} = \frac{32}{x + 8}$$

$$\frac{12}{\mathbf{8} - 2} \overset{?}{=} \frac{32}{\mathbf{8} + 8}$$

$$\frac{12}{6} \overset{?}{=} \frac{32}{16}$$

$$2 = 2 \checkmark$$

Answer $x = 8$

EXAMPLE 4

The denominator of a fraction exceeds the numerator by 7. If 3 is subtracted from the numerator of the fraction and the denominator is unchanged, the value of the resulting fraction becomes $\frac{1}{3}$. Find the original fraction.

Solution Let x = the numerator of original fraction,

$x + 7$ = the denominator of the original fraction.

$\frac{x}{x+7}$ = the original fraction.

$\frac{x-3}{x+7}$ = the new fraction.

The value of the new fraction is $\frac{1}{3}$.

$$\begin{aligned} \frac{x-3}{x+7} &= \frac{1}{3} \\ 1(x+7) &= 3(x-3) \\ x + 7 &= 3x - 9 \\ \underline{-x + 9} &\quad \underline{-x + 9} \\ 16 &= 2x \\ x &= 8 \\ x + 7 &= 15 \end{aligned}$$

Check

The original fraction was $\frac{8}{15}$.

The new fraction is

$\frac{8-3}{15} = \frac{5}{15} = \frac{1}{3}$ ✔

Answer The original fraction was $\frac{8}{15}$.

EXERCISES

Writing About Mathematics

1. Jeremy said that if the means and the extremes of a proportion are interchanged, the resulting ratios form a proportion. Do you agree with Jeremy? Explain why or why not.

2. Mike said that if the same number is added to each term of a proportion, the resulting ratios form a proportion. Do you agree with Mike? Explain why or why not.

Developing Skills

In 3–8, state, in each case, whether the given ratios may form a proportion.

3. $\frac{3}{4}, \frac{30}{40}$

4. $\frac{2}{3}, \frac{10}{5}$

5. $\frac{4}{5}, \frac{16}{25}$

6. $\frac{2}{5}, \frac{5}{2}$

7. $\frac{14}{18}, \frac{28}{36}$

8. $\frac{36}{30}, \frac{18}{15}$

In 9–16, find the missing term in each proportion.

9. $\frac{1}{2} = \frac{?}{8}$
10. $\frac{3}{5} = \frac{18}{?}$
11. $\frac{1}{4} = \frac{6}{?}$
12. $\frac{4}{6} = \frac{?}{42}$
13. $4 : ? = 12 : 60$
14. $? : 9 = 35 : 63$
15. $? : 60 = 6 : 10$
16. $16 : ? = 12 : 9$

In 17–25, solve each equation and check the solution.

17. $\frac{x}{60} = \frac{3}{20}$
18. $\frac{5}{4} = \frac{x}{12}$
19. $\frac{30}{4x} = \frac{10}{24}$
20. $\frac{5}{15} = \frac{x}{x + 8}$
21. $\frac{x}{12 - x} = \frac{10}{30}$
22. $\frac{16}{8} = \frac{21 - x}{x}$
23. $\frac{3x + 3}{3} = \frac{7x - 1}{5}$
24. $12 : 15 = x : 45$
25. $5 : x + 2 = 4 : x$

In 26–28, in each case solve for x in terms of the other variables.

26. $a : b = c : x$
27. $2r : s = x : 3s$
28. $2x : m = 4r : s$

Applying Skills

In 29–36, use a proportion to solve each problem.

29. The numerator of a fraction is 8 less than the denominator of the fraction. The value of the fraction is $\frac{3}{5}$. Find the fraction.

30. The denominator of a fraction exceeds twice the numerator of the fraction by 10. The value of the fraction is $\frac{5}{12}$. Find the fraction.

31. The denominator of a fraction is 30 more than the numerator of the fraction. If 10 is added to the numerator of the fraction and the denominator is unchanged, the value of the resulting fraction becomes $\frac{3}{5}$. Find the original fraction.

32. The numerator of a certain fraction is 3 times the denominator. If the numerator is decreased by 1 and the denominator is increased by 2, the value of the resulting fraction is $\frac{5}{2}$. Find the original fraction.

33. What number must be added to both the numerator and denominator of the fraction $\frac{7}{19}$ to make the resulting fraction equal to $\frac{3}{4}$?

34. The numerator of a fraction exceeds the denominator by 3. If 3 is added to the numerator and 3 is subtracted from the denominator, the resulting fraction is equal to $\frac{5}{2}$. Find the original fraction.

35. The numerator of a fraction is 7 less than the denominator. If 3 is added to the numerator and 9 is subtracted from the denominator, the new fraction is equal to $\frac{3}{2}$. Find the original fraction.

36. Slim Johnson was usually the best free-throw shooter on his basketball team. Early in the season, however, he had made only 9 of 20 shots. By the end of the season, he had made all the additional shots he had taken, thereby ending with a season record of 3 : 4. How many additional shots had he taken?

6-5 DIRECT VARIATION

s	1	2	3
P	4	8	12

If the length of a side, s, of a square is 1 inch, then the perimeter, P, of the square is 4 inches. Also, if s is 2 inches, P is 8 inches; if s is 3 inches, P is 12 inches. These pairs of values are shown in the table at the right.

From the table, we observe that, as s varies, P also varies. Comparing each value of P to its corresponding value of s, we notice that all three sets of values result in the same ratio when reduced to lowest terms:

$$\frac{P}{s} = \frac{4}{1} = \frac{8}{2} = \frac{12}{3}$$

If a relationship exists between two variables so that their ratio is a constant, that relationship between the variables is called a **direct variation**.

In every direct variation, we say that one variable **varies directly** as the other, or that one variable is **directly proportional** to the other. The constant ratio is called a **constant of variation**.

It is important to indicate the *order* in which the variables are being compared before stating the constant of variation. For example:

- In comparing P to s, $\frac{P}{s} = \frac{4}{1}$. The constant of variation is 4.
- In comparing s to P, $\frac{s}{P} = \frac{1}{4}$. The constant of variation is $\frac{1}{4}$.

Note that each proportion, $\frac{P}{s} = \frac{4}{1}$ and $\frac{s}{P} = \frac{1}{4}$, becomes $P = 4s$, the formula for the perimeter of a square.

In a direct variation, the value of each term of the ratio increases when we multiply each variable by a factor greater than 1; the value of each term of the ratio decreases when we divide each variable by a factor greater than 1, as shown below.

$$\frac{s}{P} = \frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{1 \times 3}{4 \times 3} \qquad \frac{s}{P} = \frac{1}{4} = \frac{2 \div 2}{8 \div 2} = \frac{3 \div 3}{12 \div 3}$$

EXAMPLE 1

If x varies directly as y, and $x = 1.2$ when $y = 7.2$, find the constant of variation by comparing x to y.

Solution

$$\text{Constant of variation} = \frac{x}{y} = \frac{1.2}{7.2} = \frac{1.2 \div 1.2}{7.2 \div 1.2} = \frac{1}{6}$$

Answer $\frac{1}{6}$ ◻

EXAMPLE 2

The table gives pairs of values for the variables x and y.

a. Show that one variable varies directly as the other.

b. Find the constant of variation by comparing y to x.

c. Express the relationship between the variables as a formula.

d. Find the values missing in the table.

x	1	2	3	10	?
y	8	16	24	?	1,600

Solution

a. $\frac{x}{y} = \frac{1}{8}$ $\quad \frac{x}{y} = \frac{2}{16} = \frac{2 \div 2}{16 \div 2} = \frac{1}{8}$ $\quad \frac{x}{y} = \frac{3}{24} = \frac{3 \div 3}{24 \div 3} = \frac{1}{8}$

Since all the given pairs of values have the same ratio, x and y vary directly.

b. Constant of variation $= \frac{8}{1} = 8$.

c. Write a proportion that includes both variables and the constant of variation:

$$\frac{x}{y} = \frac{1}{8}$$

$$y = 8x$$ Cross multiply to obtain the formula.

d. Substitute the known value in the equation written in **b**. Solve the equation.

When $x = 10$, find y.	When $y = 1{,}600$, find x.
$y = 8x$	$y = 8x$
$y = 8(10)$	$1{,}600 = 8x$
$y = 80$	$200 = x$

Answers **a.** The variables vary directly because the ratio of each pair is the same constant.

b. 8

c. $y = 8x$

d. When $x = 10$, $y = 80$; when $y = 1{,}600$, $x = 200$.

EXAMPLE 3

There are about 90 calories in 20 grams of a cheese. Reggie ate 70 grams of this cheese. About how many calories were there in the cheese she ate if the number of calories varies directly as the weight of the cheese?

Solution Let x = number of calories in 70 grams of cheese.

$$\frac{\text{number of calories}}{\text{number of grams of cheese}} = \frac{90}{20}$$

$$\frac{x}{70} = \frac{90}{20}$$

$$20x = 90(70)$$

$$20x = 6{,}300$$

$$x = 315$$

Answer There were about 315 calories in 70 grams of the cheese.

EXERCISES

Writing About Mathematics

1. On a cross-country trip, Natasha drives at an average speed of 65 miles per hour. She says that each day, her driving time and the distance that she travels are directly proportional. Do you agree with Natasha? Explain why or why not.

2. The cost of parking at the Center City Parking Garage is \$5.50 for the first hour or part of an hour and \$2.75 for each additional half hour or part of a half hour. The maximum cost for 24 hours is \$50. Does the cost of parking vary directly as the number of hours? Explain your answer.

Developing Skills

In 3–11, in each case one value is given for each of two variables that *vary directly*. Find the constant of variation.

3. $x = 12, y = 3$
4. $d = 120, t = 3$
5. $y = 2, z = 18$
6. $P = 12.8, s = 3.2$
7. $t = 12, n = 8$
8. $I = 51, t = 6$
9. $s = 88, t = 110$
10. $A = 212, P = 200$
11. $r = 87, s = 58$

In 12–17, tell, in each case, whether one variable varies directly as the other. If it does, express the relation between the variables by means of a formula.

12.

P	3	6	9
s	1	2	3

13.

n	3	4	5
c	6	8	10

14.

x	4	5	6
y	6	8	10

15.

t	1	2	3
d	20	40	60

16.

x	2	3	4
y	−6	−9	−12

17.

x	1	2	3
y	1	4	9

In 18–20, in each case one variable varies directly as the other. Write the formula that relates the variables and find the missing numbers.

18.

h	1	2	?
A	5	?	25

19.

h	4	8	?
S	6	?	15

20.

l	2	8	?
w	1	?	7

In 21–24, state whether the relation between the variables in each equation is a direct variation. In each case, give a reason for your answer.

21. $R + T = 80$ **22.** $15T = D$ **23.** $\frac{e}{i} = 20$ **24.** $bh = 36$

25. $C = 7n$ is a formula for the cost of n articles that sell for $7 each.

a. How do C and n vary?

b. How will the cost of nine articles compare with the cost of three articles?

c. If n is doubled, what change takes place in C?

26. $A = 12l$ is a formula for the area of any rectangle whose width is 12.

a. Describe how A and l vary.

b. How will the area of a rectangle whose length is 8 inches compare with the area of a rectangle whose length is 4 inches?

c. If l is tripled, what change takes place in A?

27. The variable d varies directly as t. If $d = 520$ when $t = 13$, find d when $t = 9$.

28. Y varies directly as x. If $Y = 35$ when $x = -5$, find Y when $x = -20$.

29. A varies directly as h. $A = 48$ when $h = 4$. Find h when $A = 36$.

30. N varies directly as d. $N = 10$ when $d = 8$. Find N when $d = 12$.

Applying Skills

In 31–48, the quantities vary directly. Solve algebraically.

31. If 3 pounds of apples cost $0.89, what is the cost of 15 pounds of apples at the same rate?

32. If four tickets to a show cost $17.60, what is the cost of seven such tickets?

33. If $\frac{1}{2}$ pound of meat sells for $3.50, how much meat can be bought for $8.75?

34. Willis scores an average of 7 foul shots in every 10 attempts. At the same rate, how many shots would he score in 200 attempts?

35. There are about 60 calories in 30 grams of canned salmon. About how many calories are there in a 210-gram can?

36. There are 81 calories in a slice of bread that weighs 30 grams. How many calories are there in a loaf of this bread that weighs 600 grams?

37. There are about 17 calories in three medium-size shelled peanuts. Joan ate 30 such peanuts. How many calories were there in the peanuts she ate?

38. A train traveled 90 miles in $1\frac{1}{2}$ hours. At the same rate, how long will the train take to travel 330 miles?

39. The weight of 20 meters of copper wire is 0.9 kilograms. Find the weight of 170 meters of the same wire.

40. A recipe calls for $1\frac{1}{2}$ cups of sugar for a 3-pound cake. How many cups of sugar should be used for a 5-pound cake?

41. In a certain concrete mixture, the ratio of cement to sand is 1 : 4. How many bags of cement would be used with 100 bags of sand?

42. The owner of a house that is assessed for $12,000 pays $960 in realty taxes. At the same rate, what should be the realty tax on a house assessed for $16,500?

43. The scale on a map is given as 5 centimeters to 3.5 kilometers. How far apart are two towns if the distance between these two towns on the map is 8 centimeters?

44. David received $8.75 in dividends on 25 shares of a stock. How much should Marie receive in dividends on 60 shares of the same stock?

45. A picture $3\frac{1}{4}$ inches long and $2\frac{1}{8}$ inches wide is to be enlarged so that its length will become $6\frac{1}{2}$ inches. What will be the width of the enlarged picture?

46. An 11-pound turkey costs $9.79. At this rate, find:

a. the cost of a 14.4-pound turkey, rounded to the nearest cent.

b. the cost of a 17.5-pound turkey, rounded to the nearest cent.

c. the price per pound at which the turkeys are sold.

d. the largest size turkey, to the *nearest tenth* of a pound, that can be bought for $20 or less.

47. If a man can buy p kilograms of candy for d dollars, represent the cost of n kilograms of this candy.

48. If a family consumes q liters of milk in d days, represent the amount of milk consumed in h days.

6-6 PERCENT AND PERCENTAGE PROBLEMS

Base, Rate, and Percent

Problems dealing with discounts, commissions, and taxes involve percents. A **percent**, which is a *ratio* of a number to 100, is also called a *rate*. Here, the word *rate* is treated as a comparison of a quantity to the whole. For example, 8% (read as 8 percent) is the ratio of 8 to 100, or $\frac{8}{100}$. A percent can be expressed as a fraction or as a decimal:

$$8\% = \frac{8}{100} = 0.08$$

If an item is taxed at a rate of 8%, then a $50 pair of jeans will cost an additional $4 for tax. Here, three quantities are involved.

1. The **base**, or the sum of money being taxed, is $50.
2. The **rate**, or the rate of tax, is 8% or 0.08 or $\frac{8}{100}$.
3. The **percentage**, or the amount of tax being charged, is $4.

These three related terms may be written as a proportion or as a formula:

As a proportion	*As a formula*
$\frac{\text{percentage}}{\text{base}} = \text{rate}$	base × rate = percentage
For example:	For example:
$\frac{4}{50} = \frac{8}{100}$ or $\frac{4}{50} = 0.08$	$50 \times \frac{8}{100} = 4$ or $50 \times 0.08 = 4$

Just as we have seen two ways to look at this problem involving sales tax, we will see more than one approach to every percentage problem. Note that when we calculate using percent, we always use the fraction or decimal form of the percent.

Percent of Error

When we use a measuring device such as a ruler to obtain a measurement, the accuracy and precision of the measure is dependent on the type of instrument used and the care with which it is used. **Error** is the absolute value of the difference between a value found experimentally and the true theoretical value. For example, when the length and width of a rectangle are 13 inches and 84 inches, the true length of the diagonal, found by using the Pythagorean Theorem, is 85 inches. A student drew this rectangle and, using a ruler, found the measure of the diagonal to be $84\frac{7}{8}$ inches. The error of measurement would be

$85 - 84\frac{7}{8}$ or $\frac{1}{8}$ inches. The **percent of error** or is the ratio of the error to the true value, written as a percent.

$$\textbf{Percent of error} = \frac{|\textbf{measured value} - \textbf{true value}|}{\textbf{true value}} \times 100\%$$

In the example above, the percent of error is

$$\frac{\frac{1}{8}}{85} = \frac{1}{8} \div 85 = \frac{1}{8} \times \frac{1}{85} = \frac{1}{680} \approx 0.001470588 \approx 0.15\%$$

Note: The **relative error** is simply the percent of error written as a *decimal*.

EXAMPLE 1

Find the amount of tax on a $60 radio when the tax rate is 8%.

Solution

METHOD 1 Use the proportion: $\frac{\text{percentage}}{\text{base}} = \text{rate}$.

Let t = the percentage or amount of tax.

$$\begin{aligned} \frac{\text{amount of tax}}{\text{base}} &= \frac{8}{100} \\ \frac{t}{60} &= \frac{8}{100} \\ 100t &= 480 \\ t &= 4.80 \end{aligned}$$

The tax is $4.80.

METHOD 2 Use the formula: base × rate = percentage.

Let t = percentage or amount of tax.

Change 8% to a fraction.	Change 8% to a decimal.
base × rate = percentage	base × rate = percentage
$60 \times 8\% = t$	$60 \times 8\% = t$
$60 \times \frac{8}{100} = t$	$60 \times 0.08 = t$
$\frac{480}{100} = t$	$4.8 = t$
$4.8 = t$	

Whether the fraction or the decimal form of 8% is used, the tax is $4.80.

Answer The tax is $4.80.

EXAMPLE 2

During a sale, a store offers a discount of 25% off any purchase. What is the regular price of a dress that a customer purchased for \$73.50?

Solution The rate of the discount is 25%.

Therefore the customer paid (100 − 25)% or 75% of the regular price.

The percentage is given as \$73.50, and the base is not known.

Let n = the regular price, or base.

METHOD 1 Use the proportion.

$$\begin{aligned} \frac{\text{percentage}}{\text{base}} &= \text{rate} \\ \frac{73.50}{n} &= \frac{75}{100} \\ 75n &= 7{,}350 \\ n &= 98 \end{aligned}$$

Check

If 25% of 98 is subtracted from 98 does the difference equal 73.50?

$$0.25 \times 98 = 24.50$$

$$98 - 24.50 = 73.50 \checkmark$$

METHOD 2 Use the formula.

$$\begin{aligned} \text{base} \times \text{rate} &= \text{percentage} \\ n \times 75\% &= 73.50 \end{aligned}$$

Use fractions

$$\begin{aligned} n \times \frac{75}{100} &= 73.50 \\ n \times \frac{75}{100} \times \frac{100}{75} &= 73.50 \times \frac{100}{75} \\ n &= 98 \end{aligned}$$

Use decimals

$$\begin{aligned} n \times 0.75 &= 73.50 \\ \frac{0.75n}{0.75} &= \frac{73.50}{0.75} \\ n &= 98 \end{aligned}$$

Answer The regular price of the dress was \$98.

Alternative Solution Let n = the regular price of the dress.

Then, $0.25n$ = the discount.

$$\underbrace{\text{The price of the dress}}_{\substack{\downarrow \\ n}}\ \underset{\substack{\downarrow \\ -}}{\text{minus}}\ \underbrace{\text{the discount}}_{\substack{\downarrow \\ 0.25n}}\ \underset{\substack{\downarrow \\ =}}{\text{is}}\ \underbrace{\text{the amount the customer paid}}_{\substack{\downarrow \\ 73.50}}.$$

$$\begin{aligned} n - 0.25n &= 73.50 \\ 1.00n - 0.25n &= 73.50 \\ 0.75n &= 73.50 \\ \frac{0.75n}{0.75} &= \frac{73.50}{0.75} \\ n &= 98 \end{aligned}$$

The check is the same as that shown for Method 1.

Answer The regular price of the dress was \$98.

Percent of Increase or Decrease

A **percent of increase or decrease** gives the ratio of the amount of increase or decrease to the original amount. A sales tax is a percent of increase on the cost of a purchase. A discount is a percent of decrease on the regular price of a purchase. To find the percent of increase or decrease, find the difference between the original amount and the new amount. The original amount is the base, the absolute value of the difference is the percentage, and the percent of increase or decrease is the rate.

$$\textbf{Percent of increase or decrease} = \frac{|\textbf{original amount} - \textbf{new amount}|}{\textbf{original amount}} \times 100\%$$

EXAMPLE 3

Last year Marisa's rent was \$600 per month. This year, her rent increased to \$630 per month. What was the percent of increase in her rent?

Solution The original rent was \$600.

The new rent was \$630.

The amount of increase was $|\$600 - \$630| = \$30$.

$$\text{Percent of increase} = \frac{30}{600} = \frac{1}{20} = 0.05$$

Change 0.05 to a percent: $0.05 = 5\%$

Answer The percent of increase is 5%.

EXAMPLE 4

A store reduced the price of a television from \$840 to \$504. What was the percent of decrease in the price of the television?

Solution

$$\begin{aligned}\text{Original price} &= \$840\\ \text{New price} &= \$504\\ \text{Amount of decrease} &= |\$840 - \$504| = \$336\\ \text{Percent of decrease} &= \tfrac{336}{840} = 0.4\end{aligned}$$

Change 0.4 to a percent: $0.4 = 40\%$

Answer The percent of decrease was 40%.

EXERCISES

Writing About Mathematics

1. Callie said that two decimal places can be used in place of the percent sign. Therefore, 3.6% can be written as 0.36. Do you agree with Callie. Explain why or why not.

2. If Ms. Edwards salary was increased by 4%, her current salary is what percent of her salary before the increase? Explain your answer.

Developing Skills

In 3–11, find each indicated percentage.

3. 2% of 36
4. 6% of 150
5. 15% of 48
6. 2.5% of 400
7. 60% of 56
8. 100% of 7.5
9. $12\frac{1}{2}$% of 128
10. $33\frac{1}{3}$% of 72
11. 150% of 18

In 12–19, find each number or base.

12. 20 is 10% of what number?
13. 64 is 80% of what number?
14. 8% of what number is 16?
15. 72 is 100% of what number?
16. 125% of what number is 45?
17. $37\frac{1}{2}$% of what number is 60?
18. $66\frac{2}{3}$% of what number is 54?
19. 3% of what number is 1.86?

In 20–27, find each percent.

20. 6 is what percent of 12?
21. 9 is what percent of 30?
22. What percent of 10 is 6?
23. What percent of 35 is 28?
24. 5 is what percent of 15?
25. 22 is what percent of 22?
26. 18 is what percent of 12?
27. 2 is what percent of 400?

Applying Skills

28. A newspaper has 80 pages. If 20 of the 80 pages are devoted to advertising, what percent of the newspaper consists of advertising?

29. A test was passed by 90% of a class. If 27 students passed the test, how many students are in the class?

30. Marie bought a dress that was marked \$24. The sales tax is 8%.

a. Find the sales tax.

b. Find the total amount Marie had to pay.

31. There were 120 planes on an airfield. If 75% of the planes took off for a flight, how many planes took off?

32. One year, the Ace Manufacturing Company made a profit of \$480,000. This represented 6% of the volume of business for the year. What was the volume of business for the year?

33. The price of a new motorcycle that Mr. Klein bought was \$5,430. Mr. Klein made a down payment of 15% of the price of the motorcycle and arranged to pay the rest in installments. How much was his down payment?

34. How much silver is in 75 kilograms of an alloy that is 8% silver?

35. In a factory, 54,650 parts were made. When they were tested, 4% were found to be defective. How many parts were good?

36. A baseball team won 9 games, which was 60% of the total number of games the team played. How many games did the team play?

37. The regular price of a sweater is \$40. The sale price of the sweater is \$34. What is the percent of decrease in the price?

38. A businessman is required to collect an 8% sales tax. One day, he collected \$280 in taxes. Find the total amount of sales he made that day.

39. A merchant sold a stereo speaker for \$150, which was 25% above its cost to her. Find the cost of the stereo speaker to the merchant.

40. Bill bought a wooden chess set at a sale. The original price was \$120; the sale price was \$90. What was the percent of decrease in the price?

41. If the sales tax on \$150 is \$7.50, what is the percent of the sales tax?

42. Mr. Taylor took a 2% discount on a bill. He paid the balance with a check for \$76.44. What was the original amount of the bill?

43. Mrs. Sims bought some stock for \$2,250 and sold the stock for \$2,520. What was the percent increase in the value of the stock?

44. When Sharon sold a vacuum cleaner for \$220, she received a commission of \$17.60. What was the rate of commission?

45. On the first day of a sale, a camera was reduced by \$8. This represented 10% of the original price. On the last day of the sale, the camera was sold for 75% of the original price. What was the final selling price of the camera?

46. The regular ticketed prices of four items at Grumbell's Clothier are as follows: coat, \$139.99; blouse, \$43.99; shoes, \$89.99; jeans, \$32.99.

a. These four items were placed on sale at 20% off the regular price. Find, correct to the nearest cent, the sale price of each of these four items.

b. Describe two different ways to find the sale prices.

47. At Relli's Natural Goods, all items are being sold today at 30% off their regular prices. However, customers must still pay an 8% tax on these items. Edie, a good-natured owner, allows each customer to choose one of two plans at this sale:

Plan 1. Deduct 30% of the cost of all items, then add 8% tax to the bill.

Plan 2. Add 8% tax to the cost of all items, then deduct 30% of this total.

Which plan if either, saves the customer more money? Explain why.

48. In early March, Phil Kalb bought shares of stocks in two different companies.

Stock ABC rose 10% in value in March, then decreased 10% in April.

Stock XYZ fell 10% in value in March, then rose 10% in April.

What percent of its original price is each of these stocks now worth?

49. A dairy sells milk in gallon containers. The containers are filled by machine and the amount of milk may vary slightly. A quality control employee selects a container at random and makes an accurate measure of the amount of milk as 16.25 cups. Find the percent of error to the nearest tenth of a percent.

50. A carpenter measures the length of a board as 50.5 centimeters. The exact measure of the length was 50.1 centimeters. Find the percent of error in the carpenter's measure to the nearest tenth of a percent.

51. A 5-pound weight is placed on a gymnasium scale. The scale dial displayed $5\frac{1}{2}$ pounds. If the scale is consistently off by the same percentage, how much does an athlete weigh, to the nearest tenth of a pound, if his weight displayed on this scale is 144 pounds?

52. Isaiah answered 80% of the questions correctly on the math midterm, and 90% of the questions correctly on the math final. Can you conclude that he answered 85% of *all* the questions correctly (the average of 80% and 90%)? Justify your answer or give a counterexample.

53. In January, Amy bought shares of stocks in two different companies. By the end of the year, shares of the first company had gone up by 12% while shares of the second company had gone up by 8%. Did Amy gain a total of $12\% + 8\% = 20\%$ in her investments? Explain why or why not.

6-7 CHANGING UNITS OF MEASURE

The weight and dimensions of a physical object are expressed in terms of units of measure. In applications, it is often necessary to change from one unit of measure to another by a process called **dimensional analysis**. To do this, we multiply by a fraction whose numerator and denominator are equal measures in two different units so that, in effect, we are multiplying by the identity element, 1. For example, since 100 centimeters and 1 meter are equal measures:

$$\frac{100\text{ cm}}{1\text{ m}} = \frac{100\text{ cm}}{100\text{ cm}} = 1$$

To change 4.25 meters to centimeters, multiply by $\frac{100\text{ cm}}{1\text{ m}}$.

$$\begin{aligned} 4.25\text{ m} &= 4.25\text{ }\cancel{\text{m}} \times \frac{100\text{ cm}}{1\text{ }\cancel{\text{m}}} \\ &= 425\text{ cm} \end{aligned}$$

$$\frac{1\text{ m}}{100\text{ cm}} = \frac{1\text{ m}}{1\text{ m}} = 1$$

To change 75 centimeters to meters, multiply by $\frac{1\text{ m}}{100\text{ cm}}$.

$$\begin{aligned} 75\text{ cm} &= 75\text{ }\cancel{\text{cm}} \times \frac{1\text{ m}}{100\text{ }\cancel{\text{cm}}} \\ &= \frac{75}{100}\text{ m} \\ &= 0.75\text{ m} \end{aligned}$$

Note that in each case, the fraction was chosen so that the given unit of measure occurred in the denominator and could be "cancelled" leaving just the unit of measure that we wanted in the result.

Sometimes it is necessary to use more than one fraction to change to the required unit. For example, if we want to change 3.26 feet to centimeters and know that 1 foot = 12 inches and that 1 inch = 2.54 centimeters, it will be necessary to first use the fraction $\frac{12\text{ in.}}{1\text{ ft}}$ to change feet to inches.

$$3.26\text{ ft} = 3.26\text{ }\cancel{\text{ft}} \times \frac{12\text{ in.}}{1\text{ }\cancel{\text{ft}}} = \frac{39.12}{1}\text{ in.} = 39.12\text{ in.}$$

Then use the fraction $\frac{2.54\text{ cm}}{1\text{ in.}}$ to change inches to centimeters

$$39.12\text{ in.} = 39.12\text{ }\cancel{\text{in.}} \times \frac{2.54\text{ cm}}{1\text{ }\cancel{\text{in.}}} = \frac{99.3648}{1}\text{ cm} = 99.3648\text{ cm}$$

This answer, rounded to the nearest tenth, can be expressed as 99.4 centimeters.

EXAMPLE 1

If there are 5,280 feet in a mile, find, to the nearest hundredth, the number of miles in 1,200 feet.

Solution

How to Proceed

(1) Write a fraction equal to 1 with the required unit in the numerator and the given unit in the denominator:

$$\frac{1\text{ mi}}{5{,}280\text{ ft}}$$

(2) Multiply the given measure by the fraction written in step 1:

$$1{,}200 \text{ ft} = 1{,}200 \cancel{\text{ft}} \times \frac{1 \text{ mi}}{5{,}280 \cancel{\text{ft}}}$$

$$= \frac{1{,}200}{5{,}280} \text{ mi}$$

$$= 0.2\overline{27} \text{ mi}$$

(3) Round the answer to the nearest hundredth:

$$1{,}200 \text{ ft} = 0.23 \text{ mi}$$

Answer 0.23 mi

EXAMPLE 2

In France, apples cost 4.25 euros per kilogram. In the United States, apples cost \$1.29 per pound. If the currency exchange rate is 0.95 euros for 1 dollar, in which country are apples more expensive?

Solution Recall that "per" indicates division, that is, 4.25 euros per kilogram can be written as $\frac{4.25 \text{ euros}}{1 \text{ kilogram}}$ and \$1.29 per pound as $\frac{1.29 \text{ dollars}}{1 \text{ pound}}$.

(1) Change euros in euros per kilogram to dollars. Use $\frac{1 \text{ dollar}}{0.95 \text{ euros}}$, a fraction equal to 1.

$$\frac{4.25 \cancel{\text{euros}}}{1 \text{ kilogram}} \times \frac{1 \text{ dollar}}{0.95 \cancel{\text{euros}}} = \frac{4.25 \text{ dollars}}{0.95 \text{ kilograms}}$$

(2) Now change kilograms in dollars per kilogram to pounds. One pound equals 0.454 kilograms. Since kilograms is in the denominator, use the fraction with kilograms in the numerator.

$$\frac{4.25 \text{ dollars}}{0.95 \cancel{\text{kilograms}}} \times \frac{0.454 \cancel{\text{kilogram}}}{1 \text{ pound}} = \frac{4.25(0.454) \text{ dollars}}{0.95(1) \text{ pounds}}$$

(3) Use a calculator for the computation.

(4) The number in the display, 2.031052632, is the cost of apples in France in dollars per pound. Round the number in the display to the nearest cent: \$2.03

(5) Compare the cost of apples in France (\$2.03 per pound) to the cost of apples in the United States (\$1.29 per pound).

Answer Apples are more expensive in France.

EXAMPLE 3

Change 60 miles per hour to feet per second.

Solution Use dimensional analysis to change the unit of measure in the given rate to the required unit of measure.

(1) Write 60 miles per hour as a fraction:

$$\frac{60\text{ mi}}{1\text{ hr}}$$

(2) Change miles to feet. Multiply by a ratio with miles in the denominator to cancel miles in the numerator:

$$= \frac{60\,\cancel{\text{mi}}}{1\text{ hr}} \times \frac{5{,}280\text{ ft}}{1\,\cancel{\text{mi}}}$$

(3) Change hours to minutes. Multiply by a ratio with hours in the numerator to cancel hours in the denominator:

$$= \frac{60(5{,}280)\text{ ft}}{1\,\cancel{\text{hr}}} \times \frac{1\,\cancel{\text{hr}}}{60\text{ min}}$$

(4) Change minutes to seconds. Multiply by a ratio with minutes in the numerator to cancel minutes in the denominator:

$$= \frac{60(5{,}280)\text{ ft}}{60\,\cancel{\text{min}}} \times \frac{1\,\cancel{\text{min}}}{60\text{ sec}}$$

(5) Compute and simplify:

$$= \frac{60(5{,}280)\text{ ft}}{60(60)\text{ sec}} = \frac{88\text{ ft}}{1\text{ sec}}$$

Alternative Solution Write the ratios in one expression and compute on a calculator.

$$\frac{60\,\cancel{\text{mi}}}{1\,\cancel{\text{hr}}} \times \frac{5{,}280\text{ ft}}{1\,\cancel{\text{mi}}} \times \frac{1\,\cancel{\text{hr}}}{60\,\cancel{\text{min}}} \times \frac{1\,\cancel{\text{min}}}{60\text{ sec}} = \frac{60(5{,}280)\text{ ft}}{60(60)\text{ sec}}$$

Answer 60 miles per hour = 88 feet per second.

EXERCISES

Writing About Mathematics

1. Sid cannot remember how many yards there are in a mile but knows that there are 5,280 feet in a mile and 3 feet in a yard. Explain how Sid can find the number of yards in a mile.

2. A recipe uses $\frac{3}{8}$ of a cup of butter. Abigail wants to use tablespoons to measure the butter and knows that 4 tablespoons equals $\frac{1}{4}$ cup. Explain how Abigail can find the number of tablespoons of butter needed for her recipe.

Developing Skills

In 3–16: **a.** write, in each case, the fraction that can be used to change the given units of measure, **b.** find the indicated unit of measure.

3. Change 27 inches to feet.

4. Change 175 centimeters to meters.

5. Change 40 ounces to pounds.

6. Change 7,920 feet to miles.

7. Change 850 millimeters to centimeters.

8. Change 12 pints to gallons.

9. Change 10.5 yards to inches

10. Change $3\frac{1}{2}$ feet to inches.

11. Change $\frac{4}{3}$ yard to feet.

12. Change 1.5 meters to centimeters.

13. Change 1.2 pounds to ounces.

14. Change 2.5 miles to feet.

15. Change 44 centimeters to millimeters.

16. Change $2\frac{1}{2}$ gallons to quarts.

Applying Skills

17. Miranda needs boards 0.8 meters long for a building project. The boards available at the local lumberyard are 2 feet, 3 feet, and 4 feet long.

a. Express the length, to the nearest hundredth of a foot, of the boards that Miranda needs to buy.

b. Which size board should Miranda buy? Explain your answer.

18. Carlos needs 24 inches of fabric for a pillow that he is making. The fabric store has a piece of material $\frac{3}{4}$ of a yard long that is already cut that he can buy for $5.50. If he has the exact size piece he needs cut from a bolt of fabric, it will cost $8.98 a yard.

a. Is the piece of material that is already cut large enough for his pillow?

b. What would be the cost of having exactly 24 inches of fabric cut?

c. Which is the better buy for Carlos?

19. A highway sign in Canada gives the speed limit as 100 kilometers per hour. Tracy is driving at 62 miles per hour. One mile is approximately equal to 1.6 kilometers.

a. Is Tracy exceeding the speed limit?

b. What is the difference between the speed limit and Tracy's speed in miles per hour?

20. Taylor has a painting for which she paid 1 million yen when she was traveling in Japan. At that time, the exchange rate was 1 dollar for 126 yen. A friend has offered her $2,000 for the painting.

a. Is the price offered larger or smaller than the purchase price?

b. If she sells the painting, what will be her profit or loss, in dollars?

c. Express the profit or loss as a percent of increase or decrease in the price of the painting.

CHAPTER SUMMARY

A **ratio**, which is a comparison of two numbers by division, is the quotient obtained when the first number is divided by a second, nonzero number. Quantities in a ratio are expressed in the same unit of measure before the quotient is found.

► **Ratio of a to b:** $\frac{a}{b}$ **or** $a : b$

A **rate** is a comparison of two quantities that may have *different* units of measure, such as a rate of speed in miles per hour. A rate that has a denominator of 1 is called a **unit rate**.

A **proportion** is an equation stating that two ratios are equal. Standard ways to write a proportion are shown below. In a proportion $a : b = c : d$, the outer terms are called the **extremes**, and the inner terms are the **means**.

► **Proportion:** $\underbrace{a : \overbrace{b = c}^{\text{means}} : d}_{\text{extremes}}$ **or** $\dfrac{\overset{\substack{\text{extreme}\\ \downarrow}}{a}}{\underset{\substack{\uparrow\\ \text{mean}}}{b}} = \dfrac{\overset{\substack{\text{mean}\\ \downarrow}}{c}}{\underset{\substack{\uparrow\\ \text{extreme}}}{d}}$

In a proportion, the product of the means is equal to the product of the extremes, or alternatively, the **cross products** are equal. This process is also called **cross-multiplication**.

A **direct variation** is a relation between two variables such that their ratio is always the same value, called the **constant of variation**. For example, the diameter of a circle is always twice the radius, so $\frac{d}{r} = 2$ shows a direct variation between d and r with a constant of variation 2.

A **percent (%)**, which is a ratio of a number to 100, is also called a **rate**. Here, the word *rate* is treated as a comparison of a quantity to a whole. In basic formulas, such as those used with discounts and taxes, the **base** and **percentage** are numbers, and the rate is a percent.

$$\frac{\text{percentage}}{\text{base}} = \text{rate} \qquad \textit{or} \qquad \text{base} \times \text{rate} = \text{percentage}$$

The **percent of error** is the ratio of the absolute value of the difference between a measured value and a true value to the true value, expressed as a percent.

$$\text{Percent of error} = \frac{|\text{measured value} - \text{true value}|}{\text{true value}} \times 100\%$$

The **relative error** is the percent of error expressed as a decimal.

The **percent of increase or decrease** is the ratio of the absolute value of the difference between the original value and the new value to the original value.

$$\text{Percent of increase or decrease} = \frac{|\text{original value} - \text{new value}|}{\text{original value}} \times 100\%$$

VOCABULARY

6-1 Ratio • Equivalent ratios • Simplest form • Continued ratio

6-2 Rate • Lowest terms • Unit rate • Unit price

6-4 Proportion • Extremes • Means • Cross-multiplying • Cross product

6-5 Direct variation • Directly proportional • Constant of variation

6-6 Percent • Base • Rate • Percentage • Error • Percent of error • Relative error • Percent increase • Percent decrease

6-7 Dimensional analysis

REVIEW EXERCISES

1. Can an 8 inch by 12 inch photograph be reduced to a 3 inch by 5 inch photograph? Explain why or why not?

2. Karen has a coupon for an additional 20% off the sale price of any dress. She wants to buy a dress that is on sale for 15% off of the original price. Will the original price of the dress be reduced by 35%? Explain why or why not.

In 3–6, express each ratio in simplest form.

3. 30 : 35

4. $8w$ to $12w$

5. $\frac{3}{8}$ to $\frac{5}{8}$

6. 75 millimeters : 15 centimeters

In 7–9, in each case solve for x and check.

7. $\frac{8}{2x} = \frac{12}{9}$

8. $\frac{x}{x + 5} = \frac{1}{2}$

9. $\frac{4}{x} = \frac{6}{x + 3}$

10. The ratio of two numbers is 1 : 4, and the sum of these numbers is 40. Find the numbers.

In 11–13, in each case, select the numeral preceding the choice that makes the statement true.

11. In a class of 9 boys and 12 girls, the ratio of the number of girls to the number of students in the class is

(1) 3 : 4 (2) 4 : 3 (3) 4 : 7 (4) 7 : 4

12. The perimeter of a triangle is 45 centimeters, and the lengths of its sides are in the ratio 2 : 3 : 4. The length of the longest side is

(1) 5 cm (2) 10 cm (3) 20 cm (4) 30 cm

13. If $a : x = b : c$, then x equals

(1) $\frac{ac}{b}$ (2) $\frac{bc}{a}$ (3) $ac - b$ (4) $bc - a$

14. Seven percent of what number is 21?

15. What percent of 36 is 45?

16. The sales tax collected on each sale varies directly as the amount of the sale. What is the constant of variation if a sales tax of $0.63 is collected on a sale of $9.00?

17. If 10 paper clips weigh 11 grams, what is the weight in grams of 150 paper clips?

18. Thelma can type 150 words in 3 minutes. At this rate, how many words can she type in 10 minutes?

19. What is the ratio of six nickels to four dimes?

20. On a stormy February day, 28% of the students enrolled at Southside High School were absent. How many students are enrolled at Southside High School if 476 students were absent?

21. After a 5-inch-by-7-inch photograph is enlarged, the shorter side of the enlargement measures 15 inches. Find the length in inches of its longer side.

22. A student who is 5 feet tall casts an 8-foot shadow. At the same time, a tree casts a 40-foot shadow. How many feet tall is the tree?

23. If four carpenters can build four tables in 4 days, how long will it take one carpenter to build one table?

24. How many girls would have to leave a room in which there are 99 girls and 1 boy in order that 98% of the remaining persons would be girls?

25. On an Australian highway, the speed limit was 110 kilometers per hour. A motorist was going 70 miles per hour. (Use 1.6 kilometers = 1 mile)

a. Should the motorist be stopped for speeding?

b. How far over or under the speed limit was the motorist traveling?

26. The speed of light is 3.00×10^5 kilometers per second. Find the speed of light in miles per hour. Use 1.61 kilometers = 1 mile. Write your answer in scientific notation with three significant digits.

27. Which offer, described in the chapter opener on page 207, is the better buy? Does the answer depend on the price of a package of frozen vegetables? Explain.

28. A proposal was made in the state senate to raise the minimum wage from \$6.75 to \$7.15 an hour. What is the proposed percent of increase to the nearest tenth of a percent?

29. In a chemistry lab, a student measured 1.0 cubic centimeters of acid to use in an experiment. The actual amount of acid that the student used was 0.95 cubic centimeters. What was the percent of error in the student's measurement? Give your answer to the nearest tenth of a percent.

Exploration

a. Mark has two saving accounts at two different banks. In the first bank with a yearly interest rate of 2%, he invests \$585. In the second bank with a yearly interest rate of 1.5%, he invests \$360. Mark claims that at the end of one year, he will make a total of 2% + 1.5% = 3.5% on his investments. Is Mark correct?

b. Two different clothing items, each costing \$30, were on sale for 10% off the ticketed price. The manager of the store claims that if you buy both items, you will save a total of 20%. Is the manager correct?

c. In a class of 505 graduating seniors, 59% were involved in some kind of after-school club and 41% played in a sport. The principal of the school claims that 59% + 41% = 100% of the graduating seniors were involved in some kind of after-school activity. Is the principal correct?

d. A certain movie is shown in two versions, the original and the director's cut. However, movie theatres can play only one of the versions. A journalist for XYZ News, reports that since 30% of theatres are showing the director's cut and 60% are showing the original, the movie is playing in 90% of all movie theatres. Is the reporter correct?

e. Based on your answers for parts **a** through **d**, write a rule stating when it makes sense to add percents.

CUMULATIVE REVIEW CHAPTERS 1–6

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. When $-4ab$ is subtracted from ab, the difference is
(1) $-3ab$ (2) -4 (3) $5ab$ (4) $-5ab$

2. If $n - 4$ represents an odd integer, the next larger odd integer is
(1) $n - 2$ (2) $n - 3$ (3) $n - 5$ (4) $n - 6$

3. The expression 2.3×10^{-3} is equal to
(1) 230 (2) 2,300 (3) 0.0023 (4) 0.023

4. The product $-2x^3y(-3xy^4)$ is equal to
(1) $-6x^3y^4$ (2) $6x^4y^5$ (3) $-6x^4y^5$ (4) $6x^3y^4$

5. What is the multiplicative inverse of $\frac{3}{2}$?
(1) $-\frac{3}{2}$ (2) $\frac{2}{3}$ (3) $-\frac{2}{3}$ (4) 1

6. If $0.2x + 4 = x - 0.6$, then x equals
(1) -46 (2) -460 (3) 5.75 (4) 0.575

7. The product $-3^4 \times 3^3 =$
(1) 3^7 (2) -3^7 (3) 9^7 (4) -9^7

8. The diameter of a circle whose area is 144π square centimeters is
(1) 24π cm (2) 6 cm (3) 12 cm (4) 24 cm

9. Jeannine paid \$88 for a jacket that was on sale for 20% off the original price. The original price of the jacket was
(1) \$105.60 (2) \$110.00 (3) \$440.00 (4) \$70.40

10. When $a = -2$ and $b = -5$, $a^2 - ab$ equals
(1) -14 (2) 14 (3) -6 (4) 6

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

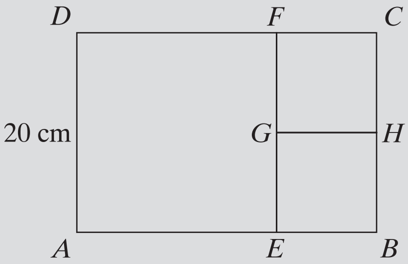
11. Find two integers, x and $x + 1$, whose squares differ by 25.

12. Solve and check: $4(2x - 1) = 5x + 5$

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Rectangle $ABCD$ is separated into three squares as shown in the diagram at the right. The length of DA is 20 centimeters.



a. Find the measure of $\overline{AB}$.

b. Find the ratio of the area of square $AEFD$ to the area of rectangle $ABCD$.

14. Sam drove a distance of 410 miles in 7 hours. For the first part of the trip his average speed was 40 miles per hour and for the remainder of the trip his average speed was 60 miles per hour. How long did he travel at each speed?

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

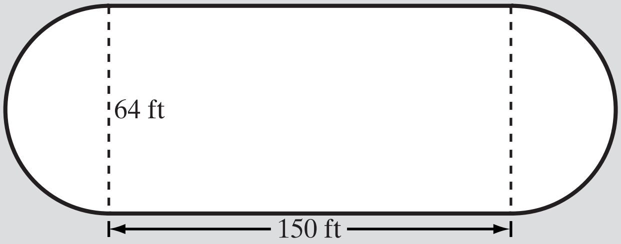
15. Three friends started a part-time business. They plan, each month, to share the profits in the ratio of the number of hours that each worked. During the first month, Rita worked 18 hours, Fred worked 30 hours, and Glen worked 12 hours.

a. Express, in lowest terms, the ratio of the times that they worked during the first month.

b. Find the amount each should receive if the profit for this first month was \$540.

c. Find the amount each should receive if the profit for this first month was \$1,270.

16. A skating rink is in the form of a rectangle with a semicircle at each end as shown in the diagram. The rectangle is 150 feet long and 64 feet wide. Scott skates around the rink 2.5 feet from the edge.



a. Scott skates once around the rink. Find, to the nearest ten feet, the distance that he skated.

b. Scott wants to skate at least 5 miles. What is the smallest number of complete trips around the rink that he must make?