1

NUMBER SYSTEMS

- The athletic department needs to transport 125 students, including the basketball team and supporters, to a playoff game. If each bus can accommodate 48 students, how many buses will be needed for the trip?

- The distance from the school to the game is 125 miles. If the bus travels at an average rate of 48 miles per hour, how long will the trip take?

- Students are having a recycling drive to help pay for the trip. One group of students collected 125 cans that will be placed in cases of 48 cans each. Only full cases can be returned to the distributor for a deposit refund. How many cases can be returned?

Each of these is a simple problem. How are the three problems alike? Why are their answers different?

In this chapter you will review the real numbers system and its subsets, use estimation skills and rational approximations to interpret calculator results, and begin to integrate the different areas of mathematics through the study of numbers, number lines, graphs, and geometric figures.

1-1 THE INTEGERS

Mathematics is the study of numbers, shapes, arrangements, relationships, and reasoning. Mathematics is both a science and an art that can be used to describe the world in which we live, to solve problems, and to create new ideas.

Numbers, which are a basic part of mathematics, help us to understand algebra, to measure geometric objects, and to make predictions using probability and statistics. In this chapter we will study numbers such as those shown below:
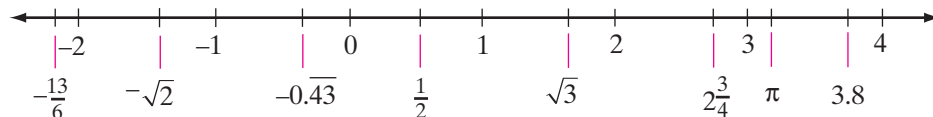
Every point on this number line corresponds to a **real number**. What are real numbers? What is meant by values such as $\sqrt{3}$ and $-0.\overline{43}$? Let us begin with simpler numbers that we know.

Symbols for Numbers

A **number** is really an idea: it is something that we can talk about and think about. We represent numbers in writing by using the symbols 1, 2, 3, 4, and so on. These symbols, called **numerals**, are not numbers but are used to represent numbers.

Counting Numbers or Natural Numbers

The **counting numbers**, which are also called **natural numbers**, are represented by the symbols

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$$

The three dots after the 12 indicate that the numbers continue *in the same pattern* without end. The smallest counting number is 1. Every counting number has a **successor** that is 1 more than that number. The successor of 1 is 2, the successor of 2 is 3, and so on. Since this process of counting is endless, there is no last counting number.

On the number line, the points associated with counting numbers are highlighted and an arrow shows that these numbers continue without end.

The Set of Whole Numbers

Zero is not a counting number. By combining 0 with all the counting numbers, we form the set of **whole numbers**. The whole numbers are represented by the symbols

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$$

The smallest whole number is 0. There is no largest whole number. Notice that the number line has been extended to include the number 0.

A **set** is a collection of distinct objects or elements. A set is usually indicated by enclosing the names or symbols for its elements within a pair of braces, { }. For example, the set of whole numbers can be written as $\{0, 1, 2, 3, 4, \ldots\}$.

Types of Sets

A **finite set** is a set whose elements can be counted. For example, the set of **digits** consists of only ten symbols, 0 through 9, that are used to write our numerals:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

An **infinite set** is a set whose elements cannot be counted because there is no end to the set. For example, the counting numbers and the whole numbers are both infinite sets.

The **empty set** or **null set** is a set that has no elements, written as {} or $\varnothing$. For example, the set of months with 32 days is empty, and the set of counting numbers between 1 and 2 is also empty.

Numerical Expressions

A **numerical expression** is a way of writing a number in symbols. The expression can be a single numeral, or it can be a collection of numerals with one or more operation symbols. For example:

$6 + 2$	$18 - 10$	4×2	$640 \div 80$
$2 \times 2 \times 2$	$2 + 2 + 2 + 2$	$1 \times 7 + 1$	8

Each of these expressions is a symbol for the number 8. In general, to **simplify** a numerical expression means to find the single number that is its value.

A calculator can be used to find the value of a numerical expression. The primary purpose of any calculator is to perform arithmetic operations, in particular, the four basic operations: addition, subtraction, multiplication, and division.

In this book, we will show the keys used on a TI-83+/84+ graphing calcula-tor. However, here, and in many of the calculator examples throughout the book, the keys listed, or similar keys, can be used on any graphing calculator.

Add 6 to the product of 3 and 9.

ENTER: 3 [×] 9 [+] 6 **ENTER**

DISPLAY:

```
3×9+6
                33
```

Answer: 33

From the quotient of 10 and 2, subtract 1.

ENTER: 10 [÷] 2 [−] 1 **ENTER**

DISPLAY:

```
10/2-1
              4
```

Answer: 4

Note that a scientific calculator uses [=] in place of **ENTER**.

The Set of Integers

The temperature on a winter day may be below 0 degrees, or someone may write a check for an amount that cannot be covered by funds in the checking account. Both situations describe **negative numbers**.

Just as the number line was extended to include 0, we can again extend it to include negative numbers. A number that is 1 less than 0 is -1; a number 2 less than 0 is -2; and so on.

Imagine that a mirror is placed at the number 0, and the counting numbers (which are thought of as *positives*) are reflected in the mirror to show *negative* numbers. Our new number line extends forever in two directions. It has no beginning and no end.

$$\longleftrightarrow$$
$$-6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6$$

Each positive number can be paired with a negative number that is the same distance from 0 but on the opposite side of 0. The numbers of each pair are called **opposites**.

- The opposite of $+1$ is -1, and the opposite of -1 is $+1$.
- The opposite of $+2$ is -2, and the opposite of -2 is $+2$, and so on.

Notice that 0 is neither positive nor negative. 0 is considered its own opposite.

The set that contains the counting numbers, 0, and the opposites of the counting numbers is the **set of integers**. The most common way to write the set of integers is to write them from the smallest to largest, in the order in which they occur on the number line. Since there is no smallest integer, the list begins with three dots to indicate that there are an infinite number of integers that are smaller than the first integer that is named. Since there is no largest integer, the list ends with three dots to indicate that there are an infinite number of integers that are larger than the last integer that is named.

$$\{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$$

Subsets of the Integers

Set A is called a **subset** of set B, written $A \subseteq B$, if every element of set A is also an element of set B. If A is a subset of B and there is at least one element in B that is not an element of A, then $A \subset B$.

Using this definition, we know that the whole numbers and the counting numbers are subsets of the integers. Counting numbers also form a subset of the whole numbers. These subsets can be illustrated in a diagram, as shown to the right.
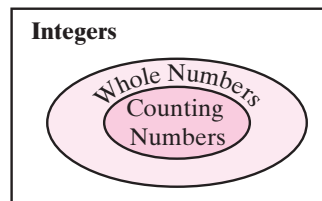
Counting numbers $\subset$ Whole numbers

Whole numbers $\subset$ Integers

Counting numbers $\subset$ Integers

There are many other subsets of the integers, such as:

1. Odd whole numbers $\{1, 3, 5, 7, 9, 11, \ldots\}$

2. Odd integers $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$

3. Even whole numbers $\{0, 2, 4, 6, 8, 10, \ldots\}$

4. Even integers $\{\ldots, -6. -4. -2, 0, 2, 4, 6, \ldots\}$

5. One-digit whole numbers $\{0, 1, 2, 3, 4, \ldots, 9\}$

Ordering the Integers

A **number line** can be used to show the numbers of a set in their relationship to each other. Each number is represented by a point on the line called the **graph** of the number.

+3
+2
+1
 0
−1
−2
−3
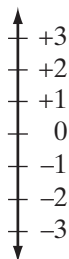
    There are two standard forms of the number line that we use. One a vertical number line (as pictured left), such as one seen on a thermometer, the higher up we go, the greater will be the number or the higher the temperature. Just as 4 is greater than 3, so is 3 greater than 0, and 0 greater than −1. It follows that −1 is greater than −2, and −2 is greater than −20.

    On a horizontal number line, positive numbers appear to the right of 0, and the negative numbers to the left of 0. The greater of any two numbers will be the one to the right and the smaller of any two numbers the one to the left. We will call this number line the **standard number line**. To build the standard number line:

1. Draw a horizontal line. Label one point on the line 0. Choose a point to the right of 0 and label it 1. The distance from 0 to 1 is called the **unit measure**.

      0 1
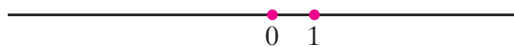
2. Place arrowheads at the ends of the line that you drew to show that this is just a part of a line that extends without end in both directions.

      0 1
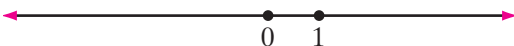
3. Use the unit measure to mark off equally spaced points to the right of 1 and label these points 2, 3, 4, and so on.

      0 1 2 3 4
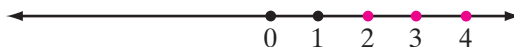
4. Use the unit measure to mark off equally spaced points to the left of 0 and label these points −1, −2, −3, and so on.

    −4 −3 −2 −1 0 1 2 3 4
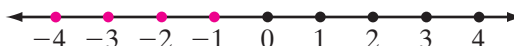
    From the number line, we see that 2 is greater than 0 and −1 is greater than −3.

Absolute Value

In every pair of nonzero opposites, the positive number is the greater. On a standard horizontal number line, the positive number is always to the right of the negative number that is its opposite. For example, 10 is greater than its opposite, −10. On a number line, 10 is to the right of −10.

The greater of a nonzero number and its opposite is called the **absolute value** of the number. The absolute value of 0 is 0.

The absolute value of a number, a, is symbolized as $|a|$. Since 10 is the greater of the two numbers 10 and -10, the absolute value of 10 is 10 and the absolute value of -10 is 10.

$$|10| = 10$$
$$|-10| = 10$$
$$|10| = |-10|$$

► **The absolute value of a positive number is the number itself; the absolute value of a negative number is the opposite of the number.**

The absolute value of a number can also be thought of as the distance between 0 and the graph of that number on the real number line. For example, $|3| = 3$, the distance between 0 and P, the graph of 3 on the real number line shown below. Also, $|-3| = 3$, the distance between 0 and S, the graph of -3 on the real number line.

$$|-3| = 3 \qquad |3| = 3$$
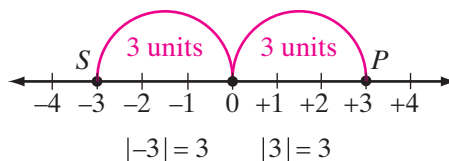
Symbols of Inequality

In our daily lives, we are often asked to compare quantities. Which is cheaper? Which weighs more? Who is taller? Which will last longer? Are two objects the same size? The answers to these questions are given by comparing quantities that are stated in numerical terms.

If two numbers are not equal, the relationship between them can be expressed as an **inequality** that can be written in several different ways.

Symbol	Example	Read
$>$	$9 > 2$	9 is greater than 2.
$<$	$2 < 9$	2 is less than 9.
$\geq$	$9 \geq 2$	9 is greater than or equal to 2.
$\leq$	$2 \leq 9$	2 is less than or equal to 9.
$\neq$	$9 \neq 2$	9 is not equal to 2.

Notice that in an inequality, the symbols $>$ and $<$ point to the smaller number.

EXAMPLE 1

Find the value of each expression.

a. $|12| + |-3|$ **b.** $|12 - 3|$

Solution   **a.** Since $|12| = 12$, and $|-3| = 3, |12| + |-3| = 12 + 3 = 15$ *Answer*

b. First, evaluate the expression inside the absolute value symbol. Then, find the absolute value.

$$|12 - 3| = |9| = 9 \quad \textit{Answer}$$

EXAMPLE 2

Tell whether each statement is true or false.

a. $-3 > -5$ *Answer:* True   **b.** $0 < -4$ *Answer:* False

c. $12 - 5 > 2$ *Answer:* True   **d.** $(2)(7) \leq 14$ *Answer:* True

EXAMPLE 3

Use the symbol $<$ to order the numbers $-4, 2,$ and -7.

Solution
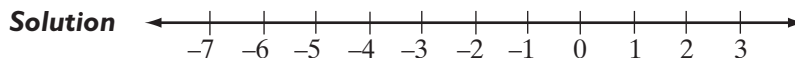
On the number line, -7 is to the left of -4 and -4 is to the left of 2. Therefore, $-7 < -4$ and $-4 < 2$.

Answer $-7 < -4 < 2$

EXAMPLE 4

Write, in each case, at least three true statements to compare the numbers in the order given.

a. 8 and 2   *Answer:* $8 > 2; 8 \neq 2; 8 \geq 2$

b. 12 and 12   *Answer:* $12 = 12; 12 \geq 12; 12 \leq 12$

EXERCISES

Writing About Mathematics

1. Olga said that the absolute value of any real number is always greater than or equal to the number. Do you agree with Olga? Explain your answer.

2. A number is represented by a and its opposite by b. If $a < b$, which letter represents a positive number and which represents a negative number. Explain your answer.

Developing Skills

In 3–12: **a.** Give the absolute value of each given number. **b.** Give another number that has the same absolute value.

3. 10.4   **4.** -7 **5.** $3 + 18$ **6.** -13 **7.** -20

8. $1\frac{1}{2}$ **9.** $-3\frac{3}{4}$ **10.** -1.45 **11.** $+2.7$ **12.** -0.02

In 13–20, state whether each sentence is true or false.

13. $|20| = 20$ **14.** $|-13| = 13$ **15.** $|-15| = -15$ **16.** $|-9| = |9|$

17. $|-7| < |7|$ **18.** $|-10| > |3|$ **19.** $|8| < |-19|$ **20.** $|-21| \geq 21$

In 21–30, find the value of each expression.

21. $|9| + |3|$ **22.** $|+8| - |+2|$ **23.** $|-6| + |4|$ **24.** $|-10| - |-5|$

25. $|4.5| - |-4.5|$ **26.** $|+6| + |-4|$ **27.** $|+6 - 4|$ **28.** $|7 - 2|$

29. $|15 - 15|$ **30.** $|-8| + |8 - 2|$

In 31–34, state whether each sentence is true or false. Give a reason for each answer.

31. $+5 > +2$ **32.** $-3 < 0$ **33.** $-7 > -1$ **34.** $-2 > -10$

In 35–40, write each inequality using the symbol $>$ or the symbol $<$.

35. $+8$ is greater than $+6$.   **36.** -8 is less than 0.

37. -5 is less than -2.   **38.** -5 is greater than -25.

39. The sum of 16 and 3 is greater than the product of 9 and 2.

40. The product of 6 and 7 is less than the quotient of 100 divided by 2.

In 41–44, express each inequality in words.

41. $+7 > -7$ **42.** $-20 < -3$ **43.** $-4 < 0$ **44.** $-9 \geq -90$

In 45–48, use the symbol $<$ to order the numbers.

45. $-4, +8, -5$ **46.** $-3, -6, +3, +6$ **47.** $+3, -2, -4, 0$ **48.** $-2, +8, 0, -8$

In 49–52, write, in each case, three true statements to compare the numbers, using the order in which they are given.

49. 8 and 14   **50.** 9 and 3   **51.** 15 and 15   **52.** 6 and −2

53. In Column I, sets of numbers are described in words. In Column II, the sets are listed using patterns and dots. Match the patterns in Column II with their correct sets in Column I.

Column I	*Column II*
1. Counting numbers	**a.** 0, 1, 2, . . . , 9
2. Whole numbers	**b.** 0, 1, 2, . . .
3. Even whole numbers	**c.** 0, 2, 4, 6, . . .
4. Odd whole numbers	**d.** 0, 2, 4, 6, 8
5. Even counting numbers	**e.** 0, 2, −2, 4, −4, 6, −6, . . .
6. Odd integers	**f.** 1, 2, 3, 4, . . .
7. Even integers	**g.** 1, 2, 3, . . . , 9
8. One-digit whole numbers	**h.** 1, 3, 5, 7, . . .
9. One-digit counting numbers	**i.** 1, 3, 5, 7, 9
10. Odd whole numbers less than 10	**j.** 2, 4, 6, 8, . . .
11. Even whole numbers less than 10	**k.** −2, −1, 0, 1, 2, 3, 4, . . .
12. Integers greater than −3	**l.** 1, −1, 3, −3, 5, −5, . . .

Applying Skills

For 54 and 55, read the problem carefully, solve the problem, and check the solution.

54. The athletic department of a school wants to transport 151 students to a basketball game. Some buses that seat 25 passengers and others that seat 34 passengers are available.

a. How many buses of each size should be scheduled for the trip so that the smallest number of buses will be used and the smallest number of seats will be empty?

b. Based on your answer to part a, how many empty seats will there be?

55. A shopkeeper has a bag of rice that he wants to divide into smaller bags. He has a container that holds 3 pounds and another that holds 4 pounds of rice. How can he use these containers to measure 5 pounds of rice?

56. Give three examples in which a negative number can be used in describing a measurement or an event.

1-2 THE RATIONAL NUMBERS

In earlier years, you worked with many numbers other than integers, such as fractions, decimals, and mixed numbers. These numbers from arithmetic, which can be located on the real number line, behave in a special way. Consider the following examples:

$$\frac{3}{5} \qquad 3\tfrac{1}{2} = \tfrac{7}{2} \qquad -\tfrac{41}{100}$$

$$0.9 = \tfrac{9}{10} \qquad 8.1 = 8\tfrac{1}{10} = \tfrac{81}{10} \qquad 0.25 = \tfrac{25}{100} = \tfrac{1}{4}$$

Each of the numbers shown here is written in the form of a fraction. In fact, every integer can be written as a fraction by writing the integer with a denominator of 1:

$$5 = \tfrac{5}{1} \qquad 0 = \tfrac{0}{1} \qquad -12 = \tfrac{-12}{1}$$

In general, any integer n can be written as $\frac{n}{1}$, which is a quotient of two integers.

The Set of Rational Numbers

The **rational numbers** are all numbers that can be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Notice that the first five letters of the word *rational* form the word *ratio*, which means a comparison of two quantities by division.

The counting numbers, the whole numbers, and the integers are all subsets of the set of rational numbers, as illustrated in the diagram.
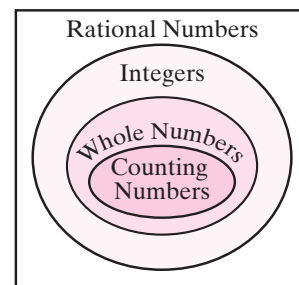
The Rational Number Line

Every rational number can be associated with a point on the real number line. For example, $\frac{1}{2}$ is midway between 0 and 1, and $-2.25 \left(\text{or } -2\tfrac{1}{4}\right)$ is one-quarter of the way between -2 and -3, closer to -2.
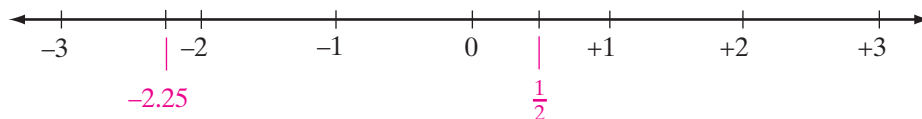
The rational numbers, like other sets studied earlier, can be ordered. In other words, given any two unequal rational numbers, we can tell which one is greater. For example, we know that $\frac{1}{2} > -1$ because, on the standard number line, $\frac{1}{2}$ is to the right of -1. There are also other ways to determine which of two rational numbers is greater, as shown in the following example.

EXAMPLE 1

Which is the greater of the numbers $\frac{7}{9}$ and $\frac{8}{11}$?

Solution   **METHOD 1.** Express the numbers as equivalent fractions with a common denominator, and compare the numerators.

$$\frac{7}{9} = \frac{7}{9} \times \frac{11}{11} = \frac{77}{99}$$
$$\frac{8}{11} = \frac{8}{11} \times \frac{9}{9} = \frac{72}{99}$$

Since $\frac{77}{99} > \frac{72}{99}$, then $\frac{7}{9} > \frac{8}{11}$.

METHOD 2. Change the fractions to decimals by dividing each numerator by its denominator to see which is greater. The answers here are from a calculator that shows ten places in each display.

ENTER: 7 ÷ 9 **ENTER**   ENTER: 8 ÷ 11 **ENTER**

DISPLAY:
```
7/9
          .7777777778
```

DISPLAY:
```
8/11
          .7272727273
```

Compare the numbers in the first two decimal places.
Since $0.77 > 0.72, \frac{7}{9} > \frac{8}{11}$.

Answer $\frac{7}{9} > \frac{8}{11}$

A Property of the Rational Numbers

► **The set of rational numbers is everywhere dense.**

In other words, given two unequal rational numbers, it is always possible to find a rational number that lies between them.

For example, some rational numbers between 1 and 2 are $1\frac{1}{2}$, $1\frac{1}{8}$, $1\frac{2}{3}$, and $1\frac{9}{10}$. In fact, there is an infinite number of rational numbers between two rational numbers.

One way to find a rational number between two rational numbers is to find their average, called the *mean*. To find the mean of two numbers, add the numbers and divide by 2. The mean (or average) of 3 and 5 is $(3 + 5) \div 2 = 8 \div 2 = 4$, a number that is between 3 and 5 on a number line. In the same way, a rational number between $\frac{1}{4}$ and $\frac{3}{4}$ can be found as follows:

$$\left(\tfrac{1}{4} + \tfrac{3}{4}\right) \div 2 = 1 \div 2 = \tfrac{1}{2}$$

A calculator can be used to do this.

ENTER: (1 ÷ 4 + 3 ÷ 4) ÷ 2 **ENTER**

DISPLAY:

```
(1/4+3/4)/2
                    .5
```

Note that, on a calculator, the rational number $\frac{1}{2}$ is written as 0.5.

Expressing a Rational Number as a Decimal

Every rational number that is not an integer can be written as a fraction. A **common fraction** is written with a numerator and denominator, for example, $\frac{3}{4}$. In a **decimal fraction** or decimal, the numerator is written after the decimal point and the denominator is indicated by the place value of the last digit. For example, the decimal fraction 0.75 has a numerator of 75 and a denominator of 100. To express as a decimal fraction a rational number named as a common fraction, we simply perform the indicated division. For example, express the following as decimals: $\frac{1}{2}, \frac{3}{4}, \frac{1}{16}$.

ENTER: 1 ÷ 2 **ENTER**

DISPLAY:

```
1/2
                    .5
```

ENTER: 3 ÷ 4 **ENTER**

DISPLAY:

```
3/4
                    .75
```

ENTER: 1 ÷ 16 **ENTER**

DISPLAY:

```
1/16
                    .0625
```

In each of the examples, $\frac{1}{2}, \frac{3}{4}$, and $\frac{1}{16}$, when we perform the division, we reach a point at which the division has no remainder, that is, a remainder of 0. If we were to continue the division with paper and pencil, we would continually obtain only zeros in the quotient. Decimals that result from such divisions, for example, 0.5, 0.75, and 0.0625, are called **terminating decimals**.

Not all rational numbers can be expressed as terminating decimals, as shown in the following examples.

Express the following as decimals: $\frac{1}{3}, \frac{2}{11}, \frac{1}{6}$.

ENTER: 1 [÷] 3 **ENTER**   ENTER: 2 [÷] 11 **ENTER**

DISPLAY: DISPLAY:

```
1/3
            .3333333333
```

```
2/11
            .1818181818
```

ENTER: 1 [÷] 6 **ENTER**

DISPLAY:

```
1/6
            .1616161617
```

In each of the above examples, when we perform the division, we find, in the quotient, that the same digit or group of digits is continually repeated in the same order. The calculator prints as many digits as possible and rounds the digit in the last decimal place that can be displayed. Decimals that keep repeating endlessly are called **repeating decimals** or **periodic decimals**.

A repeating decimal may be written in abbreviated form by placing a bar (–) over the group of digits that is to be continually repeated. For example:

$$0.333333\ldots = 0.\overline{3} \qquad 0.181818\ldots = 0.\overline{18} \qquad 0.166666\ldots = 0.1\overline{6}$$

The examples above illustrate the truth of the following statement:

► **Every rational number can be expressed as either a terminating decimal or a repeating decimal.**

Note that the equalities $0.5 = 0.5\overline{0}$ and $0.75 = 0.75\overline{0}$ illustrate the fact that every terminating decimal can be expressed as a repeating decimal that, after a point, repeats with all 0's. Then since every terminating decimal can be expressed as a repeating decimal, we will henceforth regard terminating decimals as repeating decimals. Therefore, we may say:

► **Every rational number can be expressed as a repeating decimal.**

Expressing a Decimal as a Rational Number

Use the following steps to change a terminating decimal to a fraction:

STEP 1. Read it (using place value): 0.8 is read as 8 tenths.

STEP 2. Write it as a fraction, using the same words as in step 1: $\frac{8}{10}$ is also 8 tenths

STEP 3. Reduce it (if possible): $\frac{8}{10} = \frac{4}{5}$

EXAMPLE 2

Express each decimal as a fraction:

a. 0.3   **b.** 0.37   **c.** 0.139   **d.** 0.0777

Answers   **a.** $0.3 = \frac{3}{10}$ **b.** $0.37 = \frac{37}{100}$ **c.** $0.139 = \frac{139}{1,000}$ **d.** $0.0777 = \frac{777}{10,000}$

EXERCISES

Writing About Mathematics

1. Bennie used his calculator to find the decimal value of $\frac{1}{17}$. The number in the display was 0.0588235294. Bennie knows that this is not a terminating decimal equivalent to $\frac{1}{17}$ because $\frac{588,235,294}{10,000,000,000} \neq \frac{1}{17}$. Therefore, Bennie concluded that $\frac{1}{17}$ is a rational number that is a *nonrepeating* decimal. Explain why Bennie's conclusion is incorrect.

2. Explain how you know that there is not a smallest positive rational number.

Developing Skills

In 3–12, write each rational number in the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$.

3. 0.7   **4.** 0.18   **5.** –0.21   **6.** 9   **7.** –3

8. 0   **9.** $5\frac{1}{2}$ **10.** $-3\frac{1}{3}$ **11.** 0.007   **12.** –2.3

In 13–22, state, in each case, which of the given numbers is the greater.

13. $\frac{5}{2}, \frac{7}{2}$ **14.** $-\frac{9}{3}, \frac{11}{3}$ **15.** $\frac{5}{6}, -\frac{13}{6}$ **16.** $\frac{1}{5}, -5$ **17.** $\frac{5}{2}, \frac{7}{4}$

18. $-\frac{10}{3}, -\frac{13}{6}$ **19.** $\frac{13}{6}, \frac{15}{10}$ **20.** $-\frac{5}{8}, -\frac{5}{12}$ **21.** 1.275, 1.2   **22.** $0.\overline{6}, 0.6$

In 23–32, find a rational number between each pair of given numbers.

23. 5, 6   **24.** –4, –3   **25.** –1, 0   **26.** $\frac{1}{4}, \frac{1}{2}$ **27.** $\frac{1}{2}, \frac{7}{8}$

28. $-\frac{3}{4}, -\frac{2}{3}$ **29.** –2.1, –2.2   **30.** $2\frac{1}{2}, 2\frac{5}{8}$ **31.** $-1\frac{1}{3}, -1\frac{1}{4}$ **32.** $3.05, 3\frac{1}{10}$

In 33–42, write each rational number as a repeating decimal. (Hint: Every terminating decimal has a repeating zero, for example, $0.3 = 0.3\overline{0}$.)

33. $\frac{5}{8}$ **34.** $\frac{9}{4}$ **35.** $-5\frac{1}{2}$ **36.** $\frac{13}{8}$ **37.** $-\frac{7}{12}$

38. $\frac{5}{3}$ **39.** $\frac{7}{9}$ **40.** $\frac{2}{11}$ **41.** $\frac{5}{99}$ **42.** $-\frac{5}{6}$

In 43–52, find a common fraction that names the same rational number as each decimal fraction.

43. 0.5 **44.** 0.555 **45.** -0.2 **46.** 0.12 **47.** 0.111

48. $0.125\overline{0}$ **49.** 0.2525 **50.** 0.07 **51.** 0.99875 **52.** -0.3

In 53–59, tell whether each statement is true or false, and give a reason for each answer.

53. Every integer is a rational number.

54. Whole numbers can be negative.

55. On a standard horizontal number line, the greater of two numbers is always the number farther to the right.

56. Every rational number can be written as a repeating decimal.

57. Between 0 and 1, there are an infinite number of fractions.

58. There are an infinite number of numbers between -2 and -1.

59. For every rational number, there is another rational number that is larger than the given number.

Applying Skills

For each of the following, read the problem carefully and then solve it.

60. Jacob baked some cookies. For every two cookies that he kept for his family, he gave three away to his friends. What fractional part of the cookies did he give away?

61. Margarita took part in a walk to raise money for a food pantry. After every forty-five minutes of walking, she rested for five minutes. What fractional part of the total time that she took to complete the walk was spent resting?

62. Hannah walked $\frac{3}{4}$ of the way from school to her home.

    a. What fractional part of the distance from school to her home does she have left to walk?

    b. The remaining distance is what fractional part of the distance she has already walked?

63. Josh is 72 inches tall. Ruben is $\frac{9}{10}$ as tall as Josh. John is $\frac{11}{12}$ as tall as Ruben.

    a. What is Ruben's height in inches?

    b. What fractional part of Josh's height is John?

64. Brendan has a strip of paper that is gray on the front and white on the back. The strip can be divided into three squares of the same size. He folds the paper along the diagonal of the middle square, as shown in the diagram.
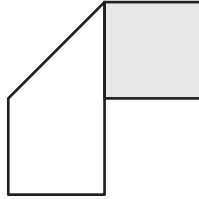
a. What fractional part of the white side of the paper is now showing?

b. What fractional part of the area showing is gray?

We have learned that on the real number line, there is one point for every rational number. We also know that there is an infinite number of rational numbers and, in turn, an infinite number of points assigned to these numbers. When we draw a number line, the dots that represent these points appear to be so dense and crowded together that the line appears to be complete. However, there are still points on the real number line that are not associated with rational numbers.

The Set of Irrational Numbers

Recall that every rational number is a repeating decimal. This includes terminating decimals, where 0 is repeated. There are infinitely many decimals, however, that do not terminate and are nonrepeating. Here is one example of such a decimal:

$$0.03003000300003000003\ldots$$

Observe that, in this number, only the digits 0 and 3 appear. First, there is a zero to the left of the decimal point, and a 0 to the right of the decimal point, then a 3 followed by two 0's, a 3 followed by three 0's, a 3 followed by four 0's and so on. If this pattern of digits continues with the number of 0's between the 3's always increasing, the number is not a repeating decimal. It is not a rational number.

An infinite, nonrepeating decimal is an **irrational number**. An irrational number cannot be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

When writing an irrational number, we use three dots (. . .) after a series of digits to indicate that the number does not terminate. The dots do *not* indicate a pattern, and no raised bar can be placed over any digits. In an irrational number, we are never certain what the next digit will be when these dots (. . .) are used.

In this section, we will see more examples of irrational numbers, both positive and negative. First, however, we need to review a few terms you learned in earlier mathematics courses.

Squares and Square Roots

To **square** a number means to multiply the number by itself. For example:

The square of 3 is 9. $3^2 = 3 \cdot 3 = 9$

The square of 4 is 16. $4^2 = 4 \cdot 4 = 16$

Calculators have a special key, $\boxed{x^2}$, that will square a number.

ENTER: 5 $\boxed{x^2}$ $\boxed{\textbf{ENTER}}$

DISPLAY:

```
5²


         25
```

To find a **square root** of a number means to find a number that, when multiplied by itself, gives the value under the **radical sign**, $\sqrt{}$. For example:

$\sqrt{9} = 3$ A square root of 9 equals 3 because $3 \cdot 3 = 9$.

$\sqrt{16} = 4$ A square root of 16 equals 4 because $4 \cdot 4 = 16$.

Calculators also have a key, $\boxed{\sqrt{}}$, that will display the square root of a number. This key is often the second function of the $\boxed{x^2}$ key. For example:

ENTER: $\boxed{\textbf{2nd}}$ $\boxed{\sqrt{}}$ 25 $\boxed{\textbf{ENTER}}$

DISPLAY:

```
√(25


          5
```

When the square root key is pressed, the calculator displays a square root sign followed by a left parenthesis. It is not necessary to close the parentheses if the entire expression that follows is under the radical sign. However, when other numbers and operations follow that are not part of the expression under the radical sign, the right parenthesis must be entered to indicate the end of the radical expression.

More Irrational Numbers

When a square measures 1 unit on every side, its diagonal measures $\sqrt{2}$ units. You can use a ruler to measure the diagonal and then show the placement of $\sqrt{2}$ on a number line.

What is the value of $\sqrt{2}$? Can we find a decimal number that, when multiplied by itself, equals 2? We expect $\sqrt{2}$ to be somewhere between 1 and 2.

Use a calculator to find the value.
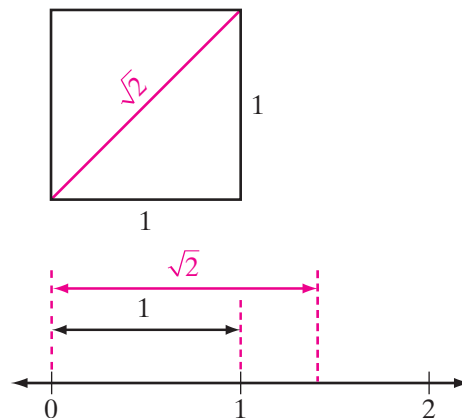
ENTER: **2nd** $\sqrt{}$ 2   **ENTER**

DISPLAY:

```
√(2
          1.414213562
```

Check this answer by multiplying:

$$1.414213562 \times 1.414213562 = 1.999999999, \text{ too small.}$$
$$1.414213563 \times 1.414213563 = 2.000000002, \text{ too large.}$$

Note that if, instead of rewriting the digits displayed on the screen, we square the answer using **2nd**   **ANS** , the graphing calculator will display 2 because in that case it uses the value of $\sqrt{2}$ that is stored in the memory of the calculator, which has more decimal places than are displayed on the screen.

No matter how many digits can be displayed on a calculator, no terminating decimal, nor any repeating decimal, can be found for $\sqrt{2}$ because

▶ $\sqrt{2}$ **is an irrational number.**

In the same way, an infinite number of square roots are irrational numbers, for example:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt{3.2} \qquad \sqrt{0.1} \qquad -\sqrt{2} \qquad -\sqrt{3}$$

The values displayed on a calculator for irrational square roots are called rational approximations. A **rational approximation** for an irrational number is a rational number that is *close to*, but *not equal to*, the value of the irrational number.

The symbol ≈ means *approximately equal to*. Therefore, it is not correct to write $\sqrt{3} = 1.732$, but it is correct to write $\sqrt{3} \approx 1.732$.

Another interesting number that you have encountered in earlier courses is π, read as "pi." Recall that π equals the circumference of a circle divided by its diameter, or $\pi = \frac{C}{d}$.
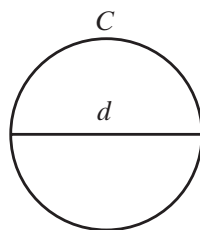
▶ **π is an irrational number.**

There are many rational approximations for π, including:

$$\pi \approx 3.14 \qquad \pi \approx \frac{22}{7} \qquad \pi \approx 3.1416$$

If π is doubled, or divided by two, or if a rational number is added to or subtracted from π, the result is again an irrational number. There are infinitely many such irrational numbers, for example:

$$2\pi \qquad \frac{\pi}{2} \qquad \pi + 7 \qquad \pi - 3$$

Approximation

Scientific calculators have a key that, when pressed, will place in the display a rational approximation for π that is more accurate than the ones given above. On a graphing calculator, when the $\boxed{\pi}$ key is accessed, the screen shows the symbol π but a rational approximation is used in the calculation.

On a graphing calculator:

ENTER: $\boxed{\text{2nd}}$ $\boxed{\pi}$ $\boxed{\text{ENTER}}$

DISPLAY:

```
π
          3.141592654
```

With a calculator, however, you must be careful how you interpret and use the information given in the display. At times, the value shown is exact, but, more often, displays that fill the screen are rational approximations. To write a rational approximation to a given number of decimal places, **round** the number.

Procedure

To round to a given decimal place:

1. Look at the digit in the place at the immediate right of the decimal place to which you are rounding the number.

2. If the digit being examined is *less than* 5, drop that digit and all digits to the right. (Example: 3.1415927 ... rounded to two decimal places is 3.14 because the digit in the third decimal place, 1, is less than 5.)

3. If the digit being examined is *greater than or equal to* 5, add 1 to the digit in the place to which you are rounding and then drop all digits to the right. (Example: 3.1415927 ... rounded to four decimal places is 3.1416 because the digit in the fifth decimal place, 9, is greater than 5.)

EXAMPLE 1

True or False: $\sqrt{5} + \sqrt{5} = \sqrt{10}$? Explain why.

Solution Use a calculator.

ENTER: [2nd] [√] 5 [ )] [+] [2nd] [√] 5 [ENTER]

DISPLAY:

```
√(5)+√(5
         4.472135955
```

ENTER: [2nd] [√] 10 [ENTER]

DISPLAY:

```
√(10
         3.16227766
```

Use these rational approximations to conclude that the values are not equal.

Answer False. $\sqrt{5} + \sqrt{5} \neq \sqrt{10}$ because $\sqrt{5} + \sqrt{5} > 4$ while $\sqrt{10} < 4$.

EXAMPLE 2

Find a rational approximation for each irrational number, to the *nearest hundredth*. **a.** $\sqrt{3}$ **b.** $\sqrt{0.1}$

Solution Use a calculator.

a. ENTER: [2nd] [√] 3 [ENTER]

DISPLAY:
```
√(3
            1.732050808
```

Use the rules for rounding. The digit in the thousandths place, 2, *is less than* 5. Drop this digit and all digits to the right of it.

Answer: $\sqrt{3} \approx 1.73$

b. ENTER: [2nd] [√] .1 [ENTER]

DISPLAY:
```
√(.1
            .316227766
```

The digit in the thousandths place, 6, *is greater than or equal to* 5. Add 1 to the digit in the hundredths place and drop all digits to the right of it.

Answer: $\sqrt{0.1} \approx 0.32$

EXAMPLE 3

The circumference C of a circle with a diameter d is found by using the formula $C = \pi d$.

a. Find the exact circumference of a circle whose diameter is 8.

b. Find, to the nearest thousandth, a rational approximation of the circumference of this circle.

Solution **a.** $C = \pi d$
$C = \pi \cdot 8$ or 8π
b. Use a calculator.

ENTER: [2nd] [π] [×] 8 [ENTER]

DISPLAY:
```
π*8
            25.13274123
```

Round the number in the display to three decimal places: 25.133.

Answers **a.** 8π is the *exact* circumference, an *irrational* number.
b. 25.133 is the *rational approximation* of the circumference, to the *nearest thousandth*.

EXAMPLE 4

Which of the following four numbers is an irrational number? In each case, the ... that follows the last digit indicates that the established pattern of digits repeats.

(1) 0.12   (3) 0.12111111 ...

(2) 0.12121212 ...   (4) 0.12112111211112 ...

Solution Each of the first three numbers is a repeating decimal. Choice (1) is a terminating decimal that can be written with a repeating zero. Choice (2) repeats the pair of digits 12 from the first decimal place and choice (3) repeats the digit 1 from the third decimal place. In choice (4), the pattern increases the number of times the digit 1 occurs after each 2. Therefore, (4) is not a repeating decimal and is irrational.

Answer (4) 0.12112111211112 ... is irrational.

EXERCISES

Writing About Mathematics

1. Erika knows that the sum of two rational numbers is always a rational number. Therefore, she concludes that the sum of two irrational numbers is always an irrational number. Give some examples that will convince Erika that she is wrong.

2. Carlos said that 3.14 is a better approximation for π than $\frac{22}{7}$. Do you agree with Carlos? Explain your answer.

Developing Skills

In 3–22, tell whether each number is rational or irrational.

3. 0.36	**4.** 0.36363636 ...	**5.** $0.3\overline{6}$	**6.** 0.363363336 ...
7. $\sqrt{8}$	**8.** 10π	**9.** 0.12131415 ...	**10.** $\sqrt{16}$
11. 0.989989998 ...	**12.** 0.725	**13.** $\sqrt{121}$	**14.** $\pi + 30$
15. -5.28	**16.** 0.14141414 ...	**17.** $-\sqrt{5}$	**18.** $-\pi$
19. $\sqrt{48}$	**20.** $\sqrt{49}$	**21.** $0.2468\overline{2}$	**22.** $\pi - 2$

23. Determine which of the following irrational numbers are between 1 and 4.

(1) $\frac{\pi}{2}$ (2) $\sqrt{5}$ (3) $\frac{\sqrt{2}}{4}$ (4) $\sqrt{11}$ (5) $-\sqrt{3}$

In 24–43 write the rational approximation of each given number:

a. as shown on a calculator display,

b. rounded to the *nearest thousandth* (three decimal places)

c. rounded to the *nearest hundredth* (two decimal places).

24. $\sqrt{5}$ **25.** $\sqrt{7}$ **26.** $\sqrt{19}$ **27.** $\sqrt{75}$ **28.** $\sqrt{63}$

29. $\sqrt{90}$ **30.** $-\sqrt{14}$ **31.** $-\sqrt{22}$ **32.** $\sqrt{0.2}$ **33.** $\sqrt{0.3}$

34. $\sqrt{12}$ **35.** $\sqrt{16}$ **36.** $\frac{\sqrt{17}}{3}$ **37.** $\frac{\pi}{3}$ **38.** $\sqrt{0.17}$

39. $-\sqrt{82}$ **40.** $\sqrt{6.5}$ **41.** $-\sqrt{55}$ **42.** $\sqrt{1,732}$ **43.** $\sqrt{241}$

44. A rational approximation for $\sqrt{3}$ is 1.732.

   a. Multiply 1.732 by 1.732.   **b.** Which is larger, $\sqrt{3}$ or 1.732?

45. a. Find $(3.162)^2$.   **b.** Find $(3.163)^2$.

   c. Is 3.162 or 3.163 a better approximation for $\sqrt{10}$? Explain why.

In 46–50, use the formula $C = \pi d$ to find, in each case, the circumference C of a circle when the diameter d is given. **a.** Write the exact value of C by using an irrational number. **b.** Find a rational approximation of C to the nearest hundredth.

46. $d = 7$ **47.** $d = 15$ **48.** $d = 72$ **49.** $d = \frac{1}{2}$ **50.** $d = 3\frac{1}{3}$

51. True or False: $\sqrt{4} + \sqrt{4} = \sqrt{8}$? Explain why or why not.

52. True or False: $\sqrt{18} + \sqrt{18} = \sqrt{36}$? Explain why or why not.

Hands-On Activity

Cut two squares, each of which measures 1 foot on each side. Cut each square along a diagonal (the line joining opposite corners of the square). Arrange the four pieces of the squares into a larger square.

a. What is the area of each of the two squares that you cut out?

b. What is the area of the larger square formed by using the pieces of the smaller squares?

c. What should be the length of each side of the larger square? Is this length rational or irrational?

d. Measure the length of each side of the larger square? Is this measurement rational or irrational?

e. Should the answers to parts **c** and **d** be the same? Explain your answer.

Recall that rational numbers can be written as repeating decimals, and that irrational numbers are decimals that do not repeat. Taken together, rational and irrational numbers make up the set of all numbers that can be written as decimals.

The **set of real numbers** is the set that consists of all rational numbers and all irrational numbers.

The accompanying diagram shows that the rational numbers are a subset of the real numbers, and the irrational numbers are also a subset of the real numbers. Notice, however, that the rationals and the irrationals take up different spaces in the diagram because they have no numbers in common. Together, these two sets of numbers form the real numbers. The cross-hatched shaded portion in the diagram contains no real numbers. The cross-hatched shading indicates that no other numbers except the rationals and irrationals are real numbers.
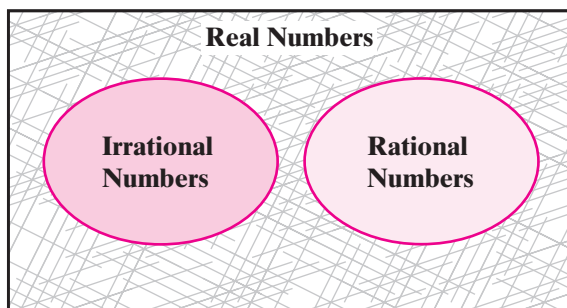
We have seen that there are an infinite number of rational numbers and an infinite number of irrationals. For every rational number, there is a corresponding point on the number line, and, for every irrational number, there is a corresponding point on the number line. All of these points, taken together, make up the **real number line**. Since there are no more holes in this line, we say that the real number line is now complete. The **completeness property of real numbers** may be stated as follows:

▶ **Every point on the real number line corresponds to a real number, and every real number corresponds to a point on the real number line.**

Ordering Real Numbers

There are two ways in which we can order real numbers:

1. Use a *number line*. On the standard horizontal real number line, the graph of the greater number is always to the right of the graph of the smaller number.

2. Use *decimals*. Given any two real numbers that are not equal, we can express them in decimal form (even using rational approximations) to see which is greater.

EXAMPLE 1

The number line that was first seen in Section 1-1 is repeated below.
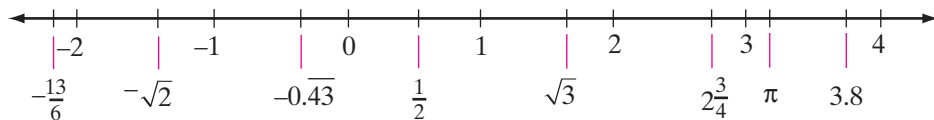
Of the numbers shown here, tell which are: **a.** counting numbers **b.** whole numbers **c.** integers **d.** rational numbers **e.** irrational numbers **f.** real numbers.

Solution **a.** Counting numbers: $1, 2, 3, 4$
b. Whole numbers: $0, 1, 2, 3, 4$
c. Integers: $-2, -1, 0, 1, 2, 3, 4$
d. Rational Numbers: $-\frac{13}{6}, -2, -1, -0.\overline{43}, 0, \frac{1}{2}, 1, 2, 2\frac{3}{4}, 3, 3.8, 4$
e. Irrational numbers $-\sqrt{2}, \sqrt{3}, \pi$
f. Real numbers: All: $-\frac{13}{6}, -2, -\sqrt{2}, -1, -0.\overline{43}, 0, \frac{1}{2}, 1, \sqrt{3}, 2, 2\frac{3}{4}, 3, \pi, 3.8, 4$

EXAMPLE 2

Order these real numbers from least to greatest, using the symbol $<$.

0.3 $\sqrt{0.3}$ $0.\overline{3}$

Solution **STEP 1.** Write each real number in decimal form:

$0.3 = 0.3000000\ldots$

$\sqrt{0.3} \approx 0.547722575$ (a rational approximation, displayed on a calculator)

$0.\overline{3} = 0.3333333\ldots$

STEP 2. Compare these decimals: $0.3000000\ldots < 0.3333333\ldots < 0.547722575$
STEP 3. Replace each decimal with the number in its original form:

$$0.3 < 0.\overline{3} < \sqrt{0.3}$$

Answer $0.3 < 0.\overline{3} < \sqrt{0.3}$

EXERCISES

Writing About Mathematics

1. There are fewer than 6 persons in my family.
   The board is less than 6 feet long.
   Each of the given statements can be designated by the inequality $x < 6$. How are the numbers that make the first statement true different from those that make the second statement true? How are they the same?

2. Dell said that it is impossible to decide whether π is larger or smaller than $\sqrt{10}$ because the calculator gives only rational approximations for these numbers. Do you agree with Dell? Explain.

3. The decimal form of a real number consists of two digits that repeat for the first one-hundred decimal places. The digits in the places that follow the one-hundredth decimal place are random, form no pattern, and do not terminate. Is the number rational or irrational? Explain.

Developing Skills

4. Twelve numbers have been placed on a number line as shown here.
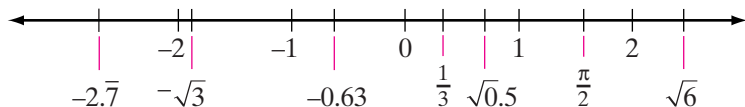
Of these numbers, tell which are:

   a. counting numbers   **b.** whole numbers   **c.** integers
   d. rational numbers   **e.** irrational numbers   **f.** real numbers

5. Given the following series of numbers:

$$\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}$$

   Of these ten numbers, tell which is (are): **a.** rational **b.** irrational **c.** real

6. Given the following series of numbers: $\pi, 2\pi, 3\pi, 4\pi, 5\pi$
   Of these five numbers, tell which is (are): **a.** rational **b.** irrational **c.** real

In 7–18, determine, for each pair, which is the greater number.

   7. 2 or 2.5   **8.** 8 or $\sqrt{8}$ **9.** $0.\overline{2}$ or 0.22   **10.** $0.\overline{2}$ or 0.23

   11. 0.7 or $0.\overline{7}$ **12.** -5.6 or -5.9 **13.** 0.43 or -0.431 **14.** $0.\overline{21}$ or $0.\overline{2}$

   15. 3.14 or π **16.** 0.5 or $\sqrt{0.5}$ **17.** $\sqrt{2}$ or 1.414   **18.** π or $\frac{22}{7}$.

In 19–24, order the numbers in each group from least to greatest by using the symbol $<$.

19. $0.202, 0.\overline{2}, 0.2022$ **20.** $0.\overline{4}, 0.45, 0.4499$ **21.** $0.\overline{67}, 0.\overline{6}, 0.\overline{667}$

22. $-\sqrt{2}, -\sqrt{3}, -1.5$ **23.** $0.5, 0.\overline{5}, \sqrt{0.3}$ **24.** $\pi, \sqrt{10}, 3.\overline{15}$

In 25–34, tell whether each statement is true or false.

25. Every real number is a rational number.

26. Every rational number is a real number.

27. Every irrational number is a real number.

28. Every real number is an irrational number.

29. Every rational number corresponds to a point on the real number line.

30. Every point on the real number line corresponds to a rational number.

31. Every irrational number corresponds to a point on the real number line.

32. Every point on the real number line corresponds to an irrational number.

33. Some numbers are both rational and irrational.

34. Every repeating decimal corresponds to a point on the real number line.

Hands-On Activity

a. Using a cloth or paper tape measure, find, as accurately as you can, the distance across and the distance around the top of a can or other object that has a circular top. If you do not have a tape measure, fit a narrow strip of paper around the circular edge and measure the length of the strip with a yardstick.

b. Divide the measure of the circumference, the distance around the circular top, by the measure of the diameter, the distance across the circular top at its center.

c. Repeat steps **a** and **b** for other circular objects and compare the quotients obtained in step **b**. Compare your results from step **b** with those of other members of your class. What conclusions can you draw?

1-5 NUMBERS AS MEASUREMENTS

In previous sections, we defined the subsets of the real numbers. When we use a counting number to identify the number of students in a class or the number of cars in the parking lot, these numbers are exact. However, to find the length of a block of wood, we must use a ruler, tape measure, or some other measuring instrument. The length that we find is dependent upon the instrument we use to measure and the care with which we make the measurement.

For example, in the diagram, a block of wood is placed along the edge of a ruler that is marked in tenths of an inch. We might say that the block of wood is 2.7 inches in length but is this measure exact?
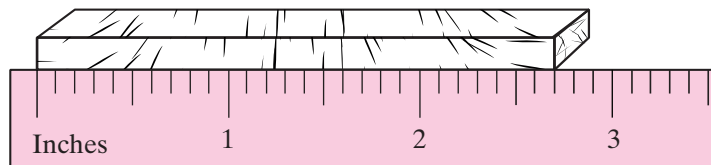
All measurements are approximate. When we say that the length of the block of wood is 2.7 inches, we mean that it is closer to 2.7 inches than it is to 2.6 inches or to 2.8 inches. Therefore, the true measure of the block of wood whose length is given as 2.7 inches is between 2.65 and 2.75 inches. In other words, the true measure is less than 0.05 inches from 2.7 and can be written as 2.7 ± 0.05 inches. The value 0.05 is called the **greatest possible error (GPE)** of measurement and is half of the place value of the last digit.

Significant Digits

The accuracy of measurement is often indicated in terms of the number of significant digits. **Significant digits** are those digits used to determine the measure and excludes those zeros that are used as place holders at the beginning of a decimal fraction and at the end of an integer.

Rules for Determining Significant Digits

RULE 1 All nonzero digits are significant.

135.6 has four significant digits. All digits are significant.

RULE 2 All zeros between significant digits are significant.

130.6 has four significant digits. The zero is significant because it is between significant digits.

RULE 3 All zeros at the end of a decimal fraction are significant.

135.000 has six significant digits. The three zeros at the end of the decimal fraction are significant.

RULE 4 Zeros that precede the first nonzero digit in a decimal fraction are not significant.

0.00424 has three significant digits. The zeros that precede the nonzero digits in the decimal fraction are placeholders and are not significant.

RULE 5 Zeros at the end of an integer may or may nor be significant. Sometimes a dot is placed over a zero if it is significant.

4,500 has two significant digits. Neither zero is significant.

4,5̇00 has three significant digits. The zero in the tens place is significant but the zero in the ones place is not.

4,50̇0 has four significant digits. The zero in the ones place is significant. Therefore, the zero in the tens place is also significant because it is between significant digits.

In any problem that uses measurement, the rules of greatest possible error and significant digits are used to determine how the answer should be stated. We can apply these rules to problems of perimeter and area. Recall the formulas for perimeter and area that you learned in previous courses.

Let P represent the perimeter of a polygon, C the circumference of a circle, and A the area of any geometric figure.

Triangle	$P = a + b + c$	$A = \frac{1}{2}bh$
Rectangle	$P = 2l + 2w$	$A = lw$
Square	$P = 4s$	$A = s^2$
Circle	$C = \pi d$ or $C = 2\pi r$	$A = \pi r^2$

Precision

The **precision** of a measurement is the place value of the last significant digit in the number. The greatest possible error of a measurement is one-half the place value of the last significant digit. In the measurement 4,500 feet, the last significant digit is in the hundreds place. Therefore, the greatest possible error is $\frac{1}{2} \times 100 = 50$. We can write the measurement as $4,500 \pm 50$ feet. One number is said to be more precise than another if the place value of its last significant digit is smaller. For example, 3.40 is more precise than 3.4 because 3.40 is correct to the nearest hundredth and 3.4 is correct to the nearest tenth.

When measures are added, the sum can be no more precise than the least precise number of the given values. For example, how should the perimeter of a triangle be stated if the measures of the sides are 34.2 inches, 27.52 inches, and 29 inches?

$$P = a + b + c$$
$$P = 34.2 + 27.52 + 29 = 90.72$$

Since the least precise measure is 29 which is precise to the nearest integer, the perimeter of the triangle should be given to the nearest integer as 91 inches.

Accuracy

The **accuracy** of a measure is the number of significant digits in the measure. One number is said to be more accurate than another if it has a larger number of significant digits. For example, 0.235 is more accurate than 0.035 because 0.235 has three significant digits and 0.035 has two, but 235 and 0.235 have the same degree of accuracy because they both have three significant digits.

   When measures are multiplied, the product can be no more accurate than the least accurate of the given values. For example, how should the area of a triangle be stated if the base measures 0.52 meters and the height measures 0.426 meters?

$$A = \tfrac{1}{2}bh$$
$$A = \tfrac{1}{2}(0.52)(0.426) = 0.5(0.52)(0.426) = 0.11076$$

Since the less accurate measure is 0.52, which has two significant digits, the area should be written with two significant digits as 0.11 square meters. Note that the $\frac{1}{2}$ or 0.5 is not a measurement but an exact value that has been determined by counting or by reasoning and therefore is not used to determine the accuracy of the answer.

   One last important note: when doing multi-step calculations, *make sure to keep at least one more significant digit in intermediate results than is needed in the final answer*. For example, if a computation requires three significant digits, then use at least four significant digits in your calculations. Otherwise, you may encounter what is known as **round-off error**, which is the phenomena that occurs when you discard information contained in the extra digit, skewing your calculations.

   In this text, you will often be asked to find the answer to an exercise in which the given numbers are thought of as exact values and the answers are given as exact values. However, in certain problems that model practical applications, when the given data are approximate measurements, you may be asked to use the precision or accuracy of the data to determine how the answer should be stated.

EXAMPLE I

State the precision and accuracy of each of the following measures.

a. 5.042 cm   **b.** 12.0 ft   **c.** 93,000,000 mi

Solution	*Precision*	*Accuracy*
a. 5.042 cm	thousandths	4 significant digits
b. 12.0 ft	tenths	3 significant digits
c. 93,000,000 mi	millions	2 significant digits

EXAMPLE 2

Of the measurements 125 feet and 6.4 feet, **a.** which is the more precise? **b.** which is the more accurate?

Solution   The measurement 125 feet is correct to the nearest foot, has an error of 0.5 feet, and has three significant digits.
    The measurement 6.4 feet is correct to the nearest tenth of a foot, has an error of 0.05 feet, and has two significant digits.

Answers **a.** The measure 6.4 feet is more precise because it has the smaller error.
b. The measure 125 feet is more accurate because it has the larger number of significant digits.

EXAMPLE 3

The length of a rectangle is 24.3 centimeters and its width is 18.76 centimeters. Using the correct number of significant digits in the answer, express **a.** the perimeter **b.** the area.

Solution **a.** Use the formula for the perimeter of a rectangle.

$$P = 2l + 2w$$
$$P = 2(24.3) + 2(18.76)$$
$$P = 86.12$$

Perimeter is a sum since $2l$ means $l + l$ and $2w$ means $w + w$. The answer should be no more precise than the least precise measurement. The least precise measurement is 24.3, given to the nearest tenth. The perimeter should be written to the nearest tenth as 86.1 centimeters.
b. To find the area of a rectangle, multiply the length by the width.

$$A = lw$$
$$A = (24.3)(18.76)$$
$$A = 455.868$$

Area is a product and the answer should be no more accurate than the least accurate of the given dimensions. Since there are three significant digits in 24.3 and four significant digits in 18.76, there should be three significant digits in the answer. Therefore, the area should be written as 456 square centimeters.

Answers **a.** 86.1 cm   **b.** 456 sq cm

EXERCISES

Writing about Mathematics

1. If 12.5 = 12.50, explain why a measure of 12.50 inches is more accurate and more precise than a measurement of 12.5 inches.

2. A circular track has a radius of 63 meters. Mario rides his bicycle around the track 10 times. Mario multiplied the radius of the track by 2π to find the circumference of the track. He said that he rode his bicycle 4.0 kilometers. Olga said that it would be more correct to say that he rode his bicycle 4 kilometers. Who is correct? Explain your answer.

Developing Skills

In 3–10, for each of the given measurements, find **a.** the accuracy **b.** the precision **c.** the error.

3. 24 in. **4.** 5.05 cm **5.** 2,400 ft **6.** 454 lb

7. 0.0012 kg **8.** 1.04 yd **9.** 1.005 m **10.** 900 mi

In 11–14, for each of the following pairs, select the measure that is **a.** the more precise **b.** the more accurate.

11. 57 in. and 4,250 in. **12.** 2.50 ft and 2.5 ft

13. 0.0003 g and 32 g **14.** 500 cm and 0.055 m

Applying Skills

In 15–18, express each answer to the correct number of significant digits.

15. Alicia made a square pen for her dog using 72.4 feet of fencing.

    a. What is the length of each side of the pen?

    b. What is the area of the pen?

16. Corinthia needed 328 feet of fencing to enclose her rectangular garden. The length of the garden is 105 feet.

    a. Find the width of the garden.

    b. Find the area of the garden.

17. Brittany is making a circular tablecloth. The diameter of the tablecloth is 10.5 inches. How much lace will she need to put along the edge of the tablecloth?

18. The label on a can of tomatoes is a rectangle whose length is the circumference of the can and whose width is the height of the can. If a can has a diameter of 7.5 centimeters and a height of 10.5 centimeters, what is the area of the label?

CHAPTER SUMMARY

A **set** is a collection of distinct objects or elements.

The **counting numbers** or **natural numbers** are $\{1, 2, 3, 4, \ldots\}$.

The **whole numbers** are $\{0, 1, 2, 3, 4, \ldots\}$.

The **integers** are $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

These sets of numbers form the basis for a **number line**, on which the length of a segment from 0 to 1 is called the **unit measure** of the line.

The **rational numbers** are all numbers that can be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Every rational number can be expressed as a repeating decimal or as a terminating decimal (which is actually a decimal in which 0 is repeated).

The **irrational numbers** are decimal numbers that do not terminate and do not repeat. On calculators and in the solution of many problems, *rational approximations* are used to show values that are close to, but not equal to, irrational numbers.

The **real numbers** consist of all rational numbers and all irrational numbers taken together. On a **real number line**, every point represents a real number and every real number is represented by a point.

The **precision** of a measurement is determined by the place value of the last significant digit. The **accuracy** of a measurement is determined by the number of significant digits in the measurement.

VOCABULARY

1-1 Mathematics • Real number • Number • Numeral • Counting numbers • Natural numbers • Successor • Whole numbers • Set • Finite set • Digit • Infinite set • Empty set • Null set • Numerical expression • Simplify • Negative numbers • Opposites • Integers • Subset • Number line • Graph • Standard number line • Unit measure • Absolute value • Inequality

1-2 Rational numbers • Everywhere dense • Common fraction • Decimal fraction • Terminating decimal • Repeating decimal • Periodic decimal

1-3 Irrational numbers • Square • Square root • Radical sign • Rational approximation • Pi (π) • Round

1-4 Real numbers • Real number line • Completeness property of real numbers

1-5 Greatest possible error (GPE) • Significant digits • Precision • Accuracy

REVIEW EXERCISES

In 1–5, use a calculator to evaluate each expression and round the result to the *nearest hundredth*.

1. 29.73×14.6 **2.** $38 \div 9$ **3.** 12.23^2 **4.** $\sqrt{216}$ **5.** $\pi \times 12$

6. Order the numbers $-5, 3,$ and -1 using the symbol $>$.

In 7–10, state whether each sentence is true of false.

7. $7 > -8$ **8.** $-7 > |-2|$ **9.** $4 < -8$ **10.** $9 \le -9$

In 11–16, write each rational number in the form $\frac{a}{b}$, where a and b are integers and $b \ne 0$.

11. 0.9 **12.** 0.45 **13.** $8\frac{1}{2}$ **14.** 14 **15.** $0.\overline{3}$ **16.** -63

17. Find a rational number between 19.9 and 20.

In 18–22, tell whether each number is rational or irrational.

18. $0.\overline{64}$ **19.** $\sqrt{6}$ **20.** $\sqrt{64}$ **21.** π

22. $0.040040004\ldots$

In 23–27, write a rational approximation of each given number: **a.** as shown on a calculator display **b.** rounded to the *nearest hundredth*.

**23.** $\sqrt{11}$ **24.** $\sqrt{0.7}$ **25.** $\sqrt{905}$ **26.** $\sqrt{1{,}599}$ **27.** π

In 28–32, determine which is the greater number in each pair.

28. 5 or $\sqrt{20}$ **29.** $|12 - 8|$ or $|12| + |-8|$ **30.** 3.2 or π

31. 0.41 or $0.\overline{4}$ **32.** $0.\overline{12}$ or 0.121

In 33–37, tell whether each statement is true or false.

33. Every integer is a real number.

34. Every rational number is an integer.

35. Every whole number is a counting number.

36. Every irrational number is a real number.

37. Between 0 and 1, there is an infinite number of rational numbers.

38. Draw a number line, showing the graphs of these numbers: $0, 1, 4, -3,$ $-1.5,$ and π.

In 39 and 40, use the given number line where the letters are equally spaced.

A   B   C   D   E   F   G   H   I   J

0   1
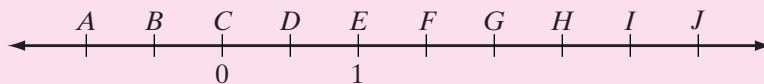
39. Find the real number that corresponds to each point indicated by a letter shown on the number line when $C = 0$ and $E = 1$.

40. Between what two consecutive points on this number line is the graph of:

  a. 1.8   **b.** -0.6 **c.** $\sqrt{2}$ **d.** π **e.** $\sqrt{6}$

41. The distance across a circular fountain (the diameter of the fountain) is 445 centimeters. The distance in centimeters around the fountain (the circumference of the fountain) can be found by multiplying 445 by π.

  a. Find the circumference of the fountain in centimeters. Round your answer to the nearest ten centimeters.

  b. When the circumference is rounded to the nearest ten centimeters, are the zeros significant?

Exploration

Using only the digits 5 and 6, and without using a radical sign:

a. Write an irrational number.

b. Write three irrational numbers that are between 5 and 6 in increasing order.

c. Write three irrational numbers that are between 0.55 and 0.56 in increasing order.

d. Write three irrational numbers that are between $0.5\overline{56}$ and $0.55\overline{6}$ in increasing order.